An attempt at defining Watanabe's propagators for $(S^1)^4$

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Slogan. Watanabe constructed an *exotic* disk bundle over S^2.

Exotic means topologically trivial, but not smoothly.

Topic (for the next 9 minutes)**.** How do we know this bundle is nontrivial?

Usually, in mathematics we work with representable functors, such as π_* or H^*. Let us consider

$$\begin{aligned} C_n(X) &\coloneqq [\{1,\dots,n\},X]\,|_{\text{embeddings}} \\ &= \{(x_1,\dots,x_n)\in X^n \mid x_i = x_j \iff i=j\} \\ &= X^n \smallsetminus \underbrace{\text{diagonals}}_{\text{collisions}} \end{aligned}$$

The $|_{\text{embeddings}}$ is not elegant so we introduce $\overline{C}_n(X)$. We allow collisions, but we remember direction – we replace diagonals with thier normal sphere bundles.

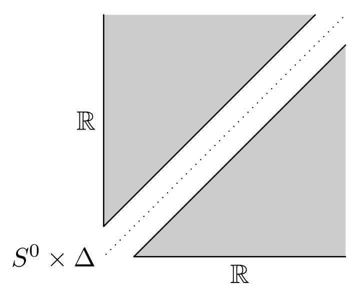

Figure 1: $\overline{C}_2(\mathbb{R})$

Example. Point 2 hits point 1. In $\overline{C}_2(X)$ point 2 is *the direction* of collision. Directions form the tangent space, hence $2 \in TX$. In other words we replaced the diagonal with its normal sphere bundle, which is isomorphic to TX by $(-v,v)\longmapsto(v,v)$.

Corollary. $C_n(-)$ can see the tangent bundle $\implies$ it is good to distinguish topological and smooth.

We have the *the configuration space*, now we can come back to the usual tools and compute its cohomology. A very *effective* way to do that is to consider the two-point collisions.

Technical detail. Configurations in a disk are the same as configurations in the sphere with ∞ forbidden. We thus write $\overline{C}_n(S^4;\infty)$.

We pull the volume form along

$$\overline{C}_n(S^4;\infty) \xrightarrow{\text{forgetful } \psi_i} \overline{C}_2(S^4;\infty) \xrightarrow[\frac{x-y}{||x-y||}]{\text{angle } \varphi} S^3$$

Graph input. We do not need to consider collisions of all pairs. Connect the points to be collided by edges.

A configuration of points gives rise to a unique graph embedding because of codimension 3.

Now for a graph Γ with n vertices we have a form:

$$\bigwedge_{i\colon \text{edges}} \underbrace{\psi_i^* \varphi_i^* \mathrm{vol}}_{\text{propagator}} \in H^3 \overline{C}_n(S^4, \infty)$$

What graph to choose? There is a theory for that called *graph cohomology.*

But I promised you bundles. We can do this in a bundle:

$$D^4 \longrightarrow E \xrightarrow{\pi} B \quad \rightsquigarrow \quad \overline{C}_n(S^4;\infty) \longrightarrow E\overline{C}_n(\pi) \longrightarrow B$$

Problem. φ depends on the coordinates.

Observe that we only care what happens on the boundary. We can require our bundle π to have a trivial subbundle

$$S^3 \longrightarrow \partial E \longrightarrow B$$

Moreover TD^4 is trivial, so we can pick a trivialisation of TD^4 over the whole trivial subbundle ∂E. We have a uniform way of defining the angle φ now. We pull by

$$\partial E\overline{C}_2(S^4;\infty) \longrightarrow S^3$$

and extend the form.

Goal. Can we do that for other manifolds? Let us start with $(S^1)^4$

Problem 1. Coordinates: already solved, $(\mathring{S}^1)^4$ is parallelizable.

Problem 2. Graph input: embedding is no longer unique. We need to label graph edges with elements of π_1. This is actually good, because we have more graphs – more forms.

Problem 3. Effectiveness: can we get any nontrivial classes that way? There is a chance.

We compute $H^3C_2(S^4;\infty)$ with the Mayer–Vietoris sequence.

$$\begin{aligned}(\mathring{S^1})^4 \times (\mathring{S^1})^4 &= \nu\Delta \cup C_2((\mathring{S^1})^4) \\ &= \nu\Delta \cap C_2(\mathring{S^1})^4 = \Delta \times S^3\end{aligned}$$

$$\begin{aligned}H^4(\Delta \times S^3) \longrightarrow H^3((\mathring{S^1})^4 \times (\mathring{S^1})^4) &\longrightarrow H^3(\nu\Delta) \oplus H^3C_2((\mathring{S^1})^4) \\ \rightsquigarrow \quad \mathbb{Z}^4 \longrightarrow \mathbb{Z}^{16} &\longrightarrow 0 \oplus \underbrace{H^3C_2((\mathring{S^1})^4)}_{\geqslant \mathbb{Z}^{12}}\end{aligned}$$

Extras (not inlcuded in the talk)**.** Cohomology computations.

$$H^*((\mathring{S^1})^4) \simeq H^*((S^1 \times S^1) \vee (S^1 \times S^1)) \simeq \mathbb{Z},\ \mathbb{Z}^4,\ \mathbb{Z}^2,\ 0,\ \ldots$$

$$H^3((\mathring{S^1})^4 \times (\mathring{S^1})^4) \simeq (\mathbb{Z}^4 \otimes \mathbb{Z}^2) \oplus (\mathbb{Z}^2 \otimes \mathbb{Z}^4) \simeq \mathbb{Z}^{16}$$
$$H^3(\nu\Delta) \simeq H^3(\Delta) \simeq 0$$
$$H^4(\Delta \times S^3) \simeq \mathbb{Z}^4 \otimes \mathbb{Z} \simeq \mathbb{Z}^4$$