# Watanabe's propagators for $(S^1)^4$

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Mapping Class Groups of Non-Simply Connected 4-Manifolds,
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# Context

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- Exotic means topologically trivial but not smoothly.
- This bundle corresponds to a nontrivial element of $\pi_1 \operatorname{Diff}(D^4, \partial)$.

## Topic for the next 9 minutes

How did Watanabe prove the nontriviality of his bundle?

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The $|_{\text{embeddings}}$ is not elegant so we introduce compactifications.

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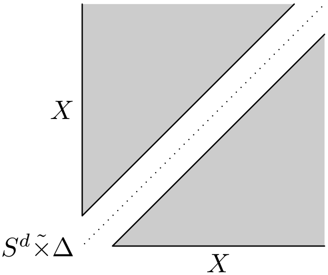

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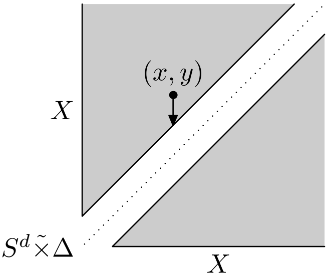


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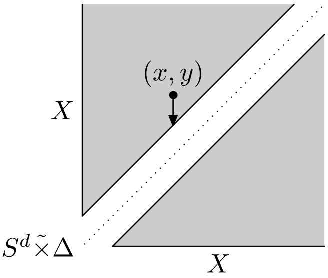


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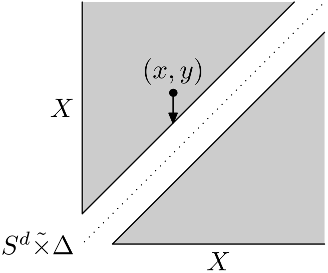


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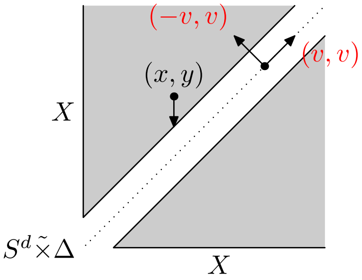


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- Directions form the tangent space, hence $y \in TX$.
- diagonal $\rightsquigarrow$ its normal sphere bundle, which is isomorphic to $TX$ by $(-v, v) \longmapsto (v, v)$.

# Corollary

$\overline{C}_n(-)$ can see the tangent bundle.

$$\Downarrow$$

It is good to distinguish topological and smooth.

# Cohomology of $C_n(\mathbb{R}^4)$

A very effective way to do that is to consider the two-point collisions. We pull the volume form along

$$\overline{C}_n(\mathbb{R}^4) \xrightarrow{\text{forgetful } \psi_i} \overline{C}_2(\mathbb{R}^4) \xrightarrow[\frac{x-y}{||x-y||}]{\text{angle } \varphi} S^3$$

Integrating forms of this type give Watanabe's invariant.

# Graph input

$$\overline{C}_n(\mathbb{R}^4) \xrightarrow{\text{forgetful } \psi_i} \overline{C}_2(\mathbb{R}^4) \xrightarrow[\frac{x-y}{||x-y||}]{\text{angle } \varphi} S^3$$

We do not need to consider all pairs. Connect the points to be collided by edges. Irrelevant how exactly.
Each graph with $n$ vertices induce a form:

$$\bigwedge_{i:\text{ edges}} \underbrace{\psi_i^* \varphi^* \mathrm{vol}}_{\text{propagator}} \in H^{3i}\overline{C}_n(\mathbb{R}^4)$$

What graph to choose? There is a theory for that, see *Kontsevich graph cohomology*.

# Back to bundles

We can do this in a bundle:

$$D^4 \longrightarrow E \xrightarrow{\pi} B \quad \rightsquigarrow \quad \overline{C}_n(\mathbb{R}^4) \longrightarrow E\overline{C}_n(\pi) \longrightarrow B$$

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Problem

$$\varphi(x, y) = \frac{x - y}{||x - y||}$$

depends on the coordinates.

## Solution to the problem

Important things happen in the boundary. We can require our bundle $\pi$ to have a trivial subbundle

$$S^3 \longrightarrow \partial E \longrightarrow B$$

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**Remark**

That is why Watanabe's bundle corresponds to something in $\mathrm{Diff}(D^4, \partial)$.

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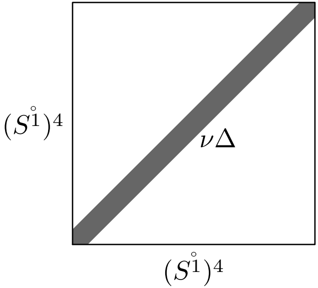
### Problem (Efectiveness)

Can we get any nontrivial classes that way? There is a chance.

# Room for propagators for $(S^1)^4$

Compute $H^3 C_2((\mathring{S}^1)^4)$ with the Mayer–Vietoris sequence.

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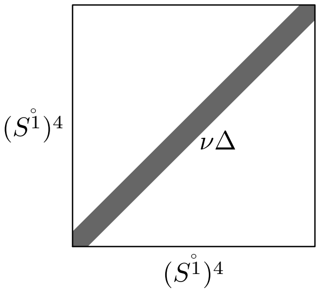


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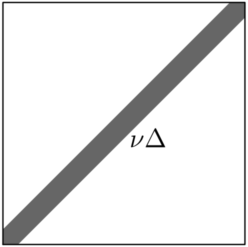


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$$
\rightsquigarrow \quad \mathbb{Z}^4 \longrightarrow \mathbb{Z}^{16} \longrightarrow 0 \oplus \underbrace{H^3 C_2((\mathring{S}^1)^4)}_{\geqslant \mathbb{Z}^{12}}
$$

Thank you for your time