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Exotic diffeomorphisms of spheres

Master's thesis
in MATHEMATICS

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Abstract

This thesis is a part of a project suggested to the author by his advisor Maciej Borodzik. The project is built on Tadayuki Watanabe's theorem that disproves the long standing Smale Conjecture in dimension four. Watanabe introduced innovative methods for studying diffeomorphism groups, whose applications are believed not to be exhausted.

In this article we focus on Watanabe's invariant $\mathcal{Z}_2$ of disk bundles over spheres. We explain the construction in detail and include a proof that it is well-defined. We provide intuition to make Watanabe's work more accessible. We hope our efforts bring the scientific community closer to understanding Watanabe's ideas.

We have a greater project in mind: the generalization of Watanabe's methods to bundles of other manifolds. The main theorem cannot be directly applied in the more general setting. However, in Section 7 we show that the proof itself is independent of the fiber being a disk. In the Appendix we present some attempts at constructing the notion of *propagator* for other manifolds and indicate directions of future research.

In Sections 1-8 we include explanations of Watanabe's results. The contribution of the author is rephrasing the results in a manner more accessible to general audience. The results in the Appendices are original part of the research of the author.

Keywords

configuration space, Kontsevich characteristic classes, exotic structures, diffeomorphism groups

Thesis domain (Socrates-Erasmus subject area codes)

11.1 Matematyka

Subject classification

57K41, 57R57, 58D27

Tytuł pracy w języku polskim

Egzotyczne dyfeomorfizmy 4-rozmaitości

Contents

1. Introduction

In 1959 Steven Smale showed that the inclusion $O(3) \hookrightarrow \mathrm{Diff}(S^2)$ is a homotopy equivalence, or in other words, that every diffeomorphism of a sphere is homotopic to a composition of symmetries and rotations of $\mathbb{R}^3$, see [Sma59]. He later conjectured that $O(n+1) \hookrightarrow \mathrm{Diff}(S^n)$ induces homotopy equivalence for arbitrary $n > 0$[♠]. In 1983 Allen Hatcher published a proof for $n = 3$, see [Hat83].

For all $n \geq 5$ the Conjecture is known to be false. To formulate the results we need some auxiliary theorems.

Theorem 1.1 ([ABK72], 1.1.5 Lemma)**.**

$$\mathrm{Diff}(S^n) \simeq \mathrm{Diff}(D^n, \partial) \times \mathrm{O}(n+1)$$

where $\mathrm{Diff}(D^n, \partial)$ is the group of diffeomorphisms of D^n that are id on some open neighborhood of the boundary.

Proof-sketch. Consider a small disk embedded in the sphere. Any diffeomorphism of the sphere is then isotopic to one that has a 'global part' – the rotation or symmetry determined by the image of the disk, and a 'local part' – what happened inside the disk. ⊠

Corollary 1.2. Finding any nontrivial element of any homotopy group of $\mathrm{Diff}(D^n, \partial)$ disproves the Smale Conjecture in dimension n.

Following, [KM63], let Θ_{n+1} denote the group of diffeomorphism classes of homotopy $(n+1)$-spheres with connected sum as multiplication.

Theorem 1.3. For $n \geq 5$ there exists a group isomorphism

$$\pi_0 \,\mathrm{Diff}(D^n, \partial) \simeq \Theta_{n+1}$$

Proof-sketch. Pick a diffeomorphism $f \colon S^n \longrightarrow S^n$ and glue two copies of D^{n+1} with f by their boundaries call the resulting manifold X. It is clear that isotopic diffeomorphisms yield the same manifold. The diffeomorphism f extends continuously to both disks, but may not extend smoothly. Therefore we have obtained a potentially exotic structure on S^{n+1}. We have constructed a homomorphism:

$$\pi_0 \,\mathrm{Diff}(S^n) \longrightarrow \Theta_{n+1}$$

which has $O(n+1)$ in its kernel, as id and $-$id extend to the whole S^n smoothly.Thus we have

$$\pi_0 \,\mathrm{Diff}(D^n, \partial) \longrightarrow \Theta_{n+1}$$

This map is bijective by Cerf's pseudoisotopy theorem, see [Cer70, Corollarie 2.]. See also [Mil56] for details of the construction. ⊠

As proven in [KM63], Θ_{n+1} is non-trivial for many n:

n	5	6	7	8	9	10	11	12	13	14	15	16
Θ_n	1	1	28	2	8	6	992	1	3	2	16256	2

[♠] according to [Hat83] the conjecture was stated in Smale's review of a paper by J. Cerf, which the author was unable to access hence many cases of the Smale conjecture are false. Falsehood of the Smale Conjecture for $n \geq 5$ has been generally known since 1980s, a good review of the results can be found in [Hata]. A theorem by Crowley and Schick that $\pi_i \operatorname{Diff}(D^n, \partial) \neq 0$ for infinitely many i, for each $n \geq 7$, proved in [CS13], disproves most cases of the Conjecture.

In 2018 Tadayuki Watanabe disproved the last open case of Smale Conjecture, for $n = 4$, by constructing elements of $\pi_1 \operatorname{Diff}(D^4, \partial)$. He has published two preprints on arXiv: one with geometric reasoning [Wat19], and one from a homological viewpoint: [Wat23], which also contains more details. It is also worth mentioning that Watanabe had previously obtained results on higher homotopy groups of diffeomorphisms groups of higher dimensional spheres obtained by similar methods to those introduced in this paper, see [Wat09a] and [Wat09b].

Here we outline the idea of Watanabe's proof. First, he constructs an invariant of smooth disk bundles over the 2-sphere[♡]. Some extra structure on said bundles is assumed along the proof, but is ultimately irrelevant for the result. Second, he construct a bundle that has a nonzero value of the invariant. Finally, standard arguments relate nontriviality of Watanabe's bundle to a family of diffeomorphisms.

This text is mostly based on the [Wat23]. We present the definition of the invariant and explain how Watanabe's bundle disproves the Smale Conjecture. Effectively we omit the construction of the bundle. We aim to give an as clear as possible explanation of the result, hence we reduce the generality to dimension 4 only. We do however have in mind extension of the methods to 4-manifolds other than spheres, so we mention some generalizations of in the proofs.

2. Graph cohomology

Definition 2.1. By *graph* we mean a finite, connected CW-complex, with every vertex of valence at least 3 (i.e., to every vertex there are at least 3 edges connected) and with no self loops (i.e., every edge connects two different vertices).

Let $\mathcal{G}^{ul}$ denote the $\mathbb{Q}$-linear space spanned formally by graphs[♣]. There is an obvious bigrading on $\mathcal{G}^{ul}$: by number of edges e and number of vertices v. We change it to better reflect the combinatorial structure that we introduce in a while. Let $k := e - v$ and $l := 2e - 3v$, called *degree* and *excess*. Let $P_k\mathcal{G}^{ul}_l$ denote the (k, l)-th bidegree, $P_k\mathcal{G}^{ul} := \bigcup_l P_k\mathcal{G}^{ul}_l$ and $\mathcal{G}^{ul}_l := \bigcup_k P_k\mathcal{G}^{ul}_l$. Note that

$$\begin{aligned} \mathbb{Z}^2 &\longrightarrow \mathbb{Z}^2 \\ (e, v) &\longmapsto (l, k) \end{aligned}$$

is $1 - 1$. Also note that if k or l is negative then $P_k\mathcal{G}^{ul}_l = 0$ and finally note that $\mathcal{G}^{ul}_0$ is exactly the set of trivalent graphs. We then label the edges by numbers $\{1 \dots e\}$ and divide by the AS[◇] relation:

$$\text{AS:} \quad \Gamma \sim \Gamma' \quad \iff \quad \Gamma \text{ and } \Gamma' \text{ differ by an odd permutation of labels}$$

Observe that it kills some of the graphs: those that have a CW-automorphism acting as an even permutation of labels. Graphs with multiple edges form a simple class of examples. Also note that any unlabeled graph can be labeled in an essentially unique way up to sign, thus we usually skip the labeling on drawings. The idea behind AS is

[♡] In fact, in much grater generality, but beyond the scope of this text
[♣] *ul* stands for *unlabeled* since we will introduce edge labeling later
[◇] Stands for *anti-symmetry*.

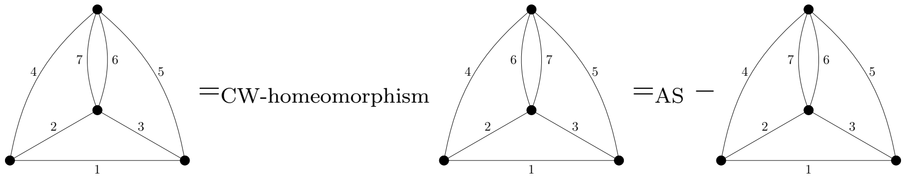


Figure 1: A graph in $P_3\mathcal{G}_1$ representing 0 due to AS

unraveled later. Each edge is going to correspond to a differential form. Compactly:

Definition 2.2. Let $\mathcal{G}$ be the $\mathbb{Q}$ space of labeled graphs bigraded by $l = 2e - 3v$ and $k = e - v$ with each bidegree denoted $P_k\mathcal{G}_l$ divided by AS.

Cohomology

On $\mathcal{G}$ we introduce differentials to make it into a chain complex. Let

$$\delta(\Gamma) = \sum_{i\colon \text{edges of } \Gamma} (-1)^{\text{label}(i)-1}\Gamma/i$$

where Γ/i denotes the graph Γ with the edge i contacted. The sign change is due to the differential form nature of edges. We want to be removing vectors of an anti-symmetric space from the first position only, so when we remove the edge i we first need to perform a number of sign-changing permutations. Observe that contracting two different edges in two different orders induces opposite labelings on the resulting graph hence $\delta^2 = 0$. Moreover, collapsing an edge shifts the l-grading by $+1$ making $(\mathcal{G}, \delta)$ into a cochain complex. Let

$$H^l(\mathcal{G}) = \frac{\ker \delta\colon \mathcal{G}_l \longrightarrow \mathcal{G}_{l+1}}{\operatorname{im} \delta\colon \mathcal{G}_{l-1} \longrightarrow \mathcal{G}_l}$$

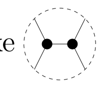
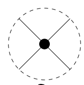
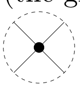
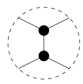
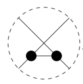
As δ does not change the k-grading we have a family of chain complexes and it makes sense to write $P_kH^l(\mathcal{G}) = H^l(P_k\mathcal{G})$.

On $\mathcal{G}$ we have a canonical basis: the graphs themselves with coefficients 1, and the standard scalar product which enables us to identify $\mathcal{G}_l$ with $\mathcal{G}_l^*$. Define $\delta^* : \mathcal{G}_l \longrightarrow \mathcal{G}_{l-1}$ by a matrix – the transposed matrix of δ in this standard basis. Then let

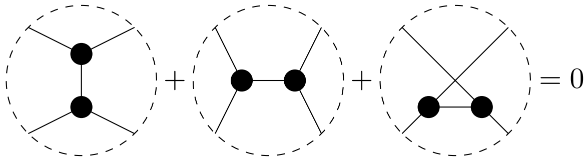
$$H_l(\mathcal{G}) \coloneqq \frac{\ker \delta^*\colon \mathcal{G}_l \longrightarrow \mathcal{G}_{l-1}}{\operatorname{im} \delta^*\colon \mathcal{G}_{l+1} \longrightarrow \mathcal{G}_l}$$

The most interesting is the $l = 0$ degree – it deserves separate name:

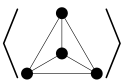
$$\mathcal{A}_k \coloneqq P_kH_0(\mathcal{G}) = P_k\mathcal{G}_0/\delta^*(\mathcal{G}_1)$$

$\delta^*(\mathcal{G}_1)$ can be computed explicitly. Collapsing an edge locally looks like  $\xrightarrow{\delta}$  (the graph outside the circle stays untouched). Other graphs that are mapped by δ to  look locally like  and . It is a choice of connecting four antennas to two vertices in a trivalent way so there are $\binom{4}{2} = 6$ options but the two vertices can always be swapped by a CW-homeomorphism so we are left with three possibilities.

As δ^* is $\delta^\intercal$ we look at their sum which has to be zero. Hence $\mathcal{A}_k = P_k\mathcal{G}_0/\operatorname{IHX}$ with IHX being:



Using this relation $\mathcal{A}_k$ can be computed by a brute-force as we know the number of edges and vertices for each k. In particular

$\mathcal{A}_2 \simeq \mathbb{Q}\Big\langle$  $\Big\rangle$

(this graph is often referred to as K_4 or W_4, although we will always use the graphical symbol).

Special element $\tilde{\zeta}_k$

We now introduce a special element behaving like $(1,\dots,1)$ that can be projected to any basis vector, thus it somewhat encodes the whole space.

Let $\mathcal{L}_k$ and $\mathcal{L}'_k$ denote the distinguished bases of $P_k\mathcal{G}_0$ and $P_k\mathcal{G}_1$. Let

$$\zeta_k \coloneqq \sum_{\Gamma\in\mathcal{L}_k} \Gamma\otimes\Gamma \in P_k\mathcal{G}_0\otimes P_k\mathcal{G}_0$$

The encoding of the whole space works as follows:

$$\begin{aligned}\forall_{\gamma\in P_k\mathcal{G}_0}\exists \left\{\substack{\text{linear functional}\\ \\ W_\gamma\colon P_k\mathcal{G}_0\longrightarrow\mathbb{Q}}\right\} \text{ s.t. } \gamma &= (W_\gamma\otimes\operatorname{id})\zeta_k\\ &= \sum_{\Gamma\in\mathcal{L}_k} W_\gamma(\Gamma)\otimes\Gamma \overset{\langle\rangle}{\longmapsto} \sum_{\Gamma\in\mathcal{L}_k} W_\gamma(\Gamma)\Gamma\in P_k\mathcal{G}_0\end{aligned}$$

where $\langle\rangle$ denotes the evaluation map for a $\mathbb{Q}$-space V:

$$\begin{aligned}\mathbb{Q}\otimes V &\longrightarrow V\\ q\otimes v &\longmapsto qv\end{aligned}$$

Since we have the chosen bases $\mathcal{L}_k$ and $\mathcal{L}'_k$ we have chosen the isomorphisms between $P_k\mathcal{G}_0$, $P_k\mathcal{G}_1$ and their duals. We abuse notation and write $v = v^*$. With this identification we write:

$$(\operatorname{id}\otimes\delta)\zeta_k = \sum_{\Gamma\in\mathcal{L}_k}\Gamma\otimes\delta\Gamma = \sum_{\Gamma'\in\mathcal{L}'_k}\delta^*\Gamma'\otimes\Gamma'\in P_k\mathcal{G}_0\otimes P_k\mathcal{G}_1$$

as $P_k\mathcal{G}_1 \simeq \operatorname{im}\delta\oplus\ker\delta$, hence for $\Gamma\otimes\delta\Gamma\neq 0$, $\Gamma\in\frac{P_k\mathcal{G}_0}{\ker\delta}\simeq\operatorname{im}\delta$, which is dual to $\operatorname{im}\delta^*\subset P_k\mathcal{G}_1$, thus $\Gamma = \delta^*\Gamma'$, for some Γ'. Then $\delta\Gamma = \delta\delta^*\Gamma = \Gamma$. Note we have silently restricted δ and δ^* to the k-th grading.

With that identification we compute:

$$
\begin{aligned}
[\gamma] \in P_k H^0(\mathcal{G}) \iff 0 &= \delta\gamma \\
&= \delta\Big\langle (W_\gamma \otimes \mathrm{id})\zeta_k \Big\rangle \\
&= \delta\Big\langle (\mathrm{id} \otimes \delta)(W_\gamma \otimes \mathrm{id})\zeta_k \Big\rangle \\
&= \delta\Big\langle (W_\gamma \otimes \mathrm{id})(\mathrm{id} \otimes \delta)\zeta_k \Big\rangle \\
&= \delta\Big\langle (W_\gamma \otimes \mathrm{id}) \sum_{\Gamma' \in \mathcal{L}_k} \delta^*\Gamma' \otimes \Gamma' \Big\rangle \\
&= \Big\langle \sum_{\Gamma' \in \mathcal{L}_k} W_\gamma(\delta^*\Gamma' \otimes \Gamma') \Big\rangle \\
&= \sum_{\Gamma' \in \mathcal{L}_k} W_\gamma(\delta^*\Gamma')\Gamma' = 0 \iff \forall_{\Gamma'} W(\delta^*\Gamma') = 0 \\
&\qquad\qquad\qquad\qquad \iff W_\gamma \text{ factors through} \\
&\qquad\qquad\qquad\qquad \ker \delta^* = \mathcal{A}_k \text{ as } \overline{W}_\gamma \colon \mathcal{A}_k \longrightarrow \mathbb{Q}
\end{aligned}
$$

Thus, any $[\gamma] \in P_k H^0$ is equal to $(\overline{W}_\gamma \otimes \mathrm{id})([\cdot] \otimes \mathrm{id})\zeta_k$ with suitably chosen $\overline{W}_\gamma$. We define

$$\tilde{\zeta_k} := \frac{1}{(2k)!(3k)!}([\cdot] \otimes \mathrm{id})\zeta_k \in \mathcal{A}_k \otimes P_k\mathcal{G}_0$$

which can be thought of as an element of $P_k H^0(\mathcal{G}, \mathcal{A}_k)$ (cohomology with coefficients in homology). In particular we will later make use of the fact that

$$(\mathrm{id} \otimes \delta)\tilde{\zeta_k} = 0$$

The $\frac{1}{(2k)!(3k)!}$ factor comes from permutations of edges and vertices and is there to avoid a constant later.

3. Vertically framed relative fiber bundles

In this section we develop the necessary extra structure of Watanabe's bundle. This are purely technical considerations.

Throughout this section $(X \to E \xrightarrow{\pi} B)$ is a fiber bundle. We often abuse the notation and call the bundle just π.

Definition 3.1. A pointed fiber bundle is a fiber bundle π over a pointed space $(B, *)$ equipped with a choice of diffeomorphism: $\xi \colon X \longrightarrow \pi^{-1}(*)$. In other words, the following diagram commutes:

$$
\begin{array}{ccc}
X & \xrightarrow{\exists \xi} & E \\
\big\downarrow & & \big\downarrow{\scriptstyle \pi} \\
* & \longrightarrow & B
\end{array}
$$

Definition 3.2. An (X, A)-bundle (called *relative fiber bundle*) is a pointed bundle $X \longrightarrow E \longrightarrow B$ with fiber X and a chosen submanifold $A \subset X$ such that the inclusion over the basepoint induces a trivial subbundle of E. In other words, the following diagram commutes:

$$\begin{tikzcd}
 & & A \arrow[rr, hook, "\iota"] \arrow[dll, hook', "*\times\mathrm{id}"'] \arrow[ddr] & & X \arrow[dll, hook', "\xi"'] \arrow[ddl] \\
B \times A \arrow[rr, hook, "\exists\varphi"] \arrow[ddr, "\mathrm{pr}_1"'] & & E \arrow[ddl, "\pi"] & & \\
 & & & * \arrow[dll, hook'] & \\
 & B & & &
\end{tikzcd}$$

Alternatively, we consider π as a $\mathrm{Diff}(X)$-bundle, then restrict the structure group to $\mathrm{Diff}(X, A)$ – diffeomorphisms of X fixing a neighborhood of A pointwise. Such bundles (their associated principal bundles to be precise) are classified by $\mathrm{B}\,\mathrm{Diff}(X, A)$. We will be interested in D^4-bundles that have trivial $S^3 = \partial D^4$-subbundles. Their classifying space is called $\mathrm{B}\,\mathrm{Diff}(D^4, \partial)$ for short.

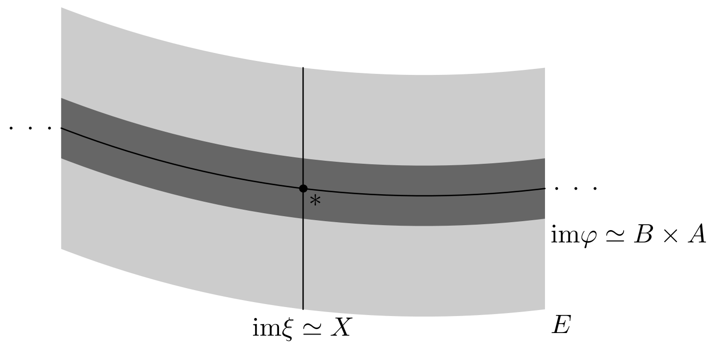


Figure 2: A schematic picture of an (X, A)-bundle $E \longrightarrow B$. Added for the reader's convenience.

We need even more structure: the vertical framing. Assume TX is trivial and pick a trivialization

$$\tau \colon TX \longrightarrow \mathbb{R}^{\dim X} \times X$$

The same can be applied to a fiber bundle. Let

$$T^v E = \ker d\pi = \bigcup_{b \in B} TX$$

and suppose it is trivial. We call this this bundle the *vertical tangent* to E. Similarly we define the *vertical boundary*:

$$\partial^v E = \bigcup_{b \in B} \partial X$$

Definition 3.3. A choice of its trivialization

$$\varepsilon \colon T^v E \xrightarrow{\simeq} \mathbb{R}^{\dim X} \times E$$

is called a *vertical framing*.

It is customary to fix one *standard* framing τ on X, and consider only those framings that agree with τ on $(\operatorname{im}\xi \cup \operatorname{im}\varphi)$. In other words, we require the vertical framing ε to satisfy the following commutative diagrams.

$$\begin{tikzcd}
TX \arrow[r, "\tau"] \arrow[d, hook, "\mathrm{d}\xi"] & \mathbb{R}^{\dim X} \times X \arrow[d, "\mathrm{id} \times \xi"] \\
T^vE \arrow[r, "\varepsilon"] & \mathbb{R}^{\dim X} \times E
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbb{R}^{\dim X} \times X \arrow[rrrr, hook, "\mathrm{id} \times \xi"] & & & & \mathbb{R}^{\dim X} \times E \\
TX \arrow[u, "\tau"] & & & & T^vE \arrow[u, "\varepsilon"'] \\
& & B \times TX \arrow[llu, "\mathrm{pr}_2"] & T^vE|_{B\times A} \arrow[l, hook'] \arrow[ru, hook, "\mathrm{d}\varphi"] &
\end{tikzcd}$$

A conventional name for a vertical framing that extends τ is τ_E. Throughout the whole paper we will only consider vertical framings for (D^4, ∂)-bundles that agree with the standard framing of D^4 outside a bounded neighborhood of zero (standard at infinity). Intuitively, we require that nothing interesting happens on the trivial subbundle. More technically, we need that so we can identify the boundary of each fiber to a point – the ∞, and identify tangent spaces to one: $T_\infty S^4$ to obtain the whole bundle TS^4. Non-standard framings could obstruct existence of bundle structure near ∞.

Note that a framing on X is a map $X \longrightarrow \mathrm{SO}(\dim X)$ to the special orthogonal group. Let

$$\mathrm{Fr}(X, A, \tau)$$

be the space of framings on X that agree with τ on A (as in the previous paragraph, standard at ∞) equipped with the compact open topology. In fact we have a homotopy equivalence:

$$\mathrm{Fr}(X, A, \tau) \simeq [(X, A), (\mathrm{SO}(\dim X), *)]$$

$\mathrm{Fr}(X, A, \tau)$ admits a left $\mathrm{Diff}(X, A)$-action by

$$f \,.\, \varepsilon = \varepsilon \circ (\mathrm{d}f)^{-1}$$

Consider the principal $\mathrm{Diff}(X, A)$-bundle:

$$\mathrm{E\,Diff}(X, A) \longrightarrow \mathrm{B\,Diff}(X, A)$$

The product bundle $\mathrm{E\,Diff}(X, A) \times \mathrm{Fr}(X, A, \tau)$ over $\mathrm{B\,Diff}(X, A)$ also admits a left $\mathrm{Diff}(X, A)$-action. The quotient

$$\widetilde{\mathrm{B\,Diff}}(X, A, \tau) := {\mathrm{E\,Diff}(X, A) \times \mathrm{Fr}(X, A, \tau)} \Big/ {\mathrm{Diff}(X, A)}$$

is still a bundle over $\mathrm{B\,Diff}(X, A)$, but with fiber $\mathrm{Fr}(X, A, \tau)$. It is the classifying space for pointed framed (X, A)-bundles, i.e., there is a bijection:

$$\left[(B, *), (\widetilde{\mathrm{B\,Diff}}(X, A, \tau), *)\right] \simeq \left\{ \begin{array}{c} \text{isomorphism classes of framed} \\ (X, A)\text{-bundles over } B \end{array} \right\}$$

The proof is simple: $\mathrm{E}\,\mathrm{Diff}(X,A)/\,\mathrm{Diff}(X,A)$ classifies (X,A)-bundles, the product with $\mathrm{Fr}(X,A,\tau)$ takes into consideration all the possible framing, but some pairs (bundle, framing) may be identified by bundle isomorphisms, hence we extend the action of $\mathrm{Diff}(X,A)$.

Moreover, for $(X,A)=(D^4,\partial)$ we have homotopy equivalence with the unpointed (!) map space:

$$\mathrm{Fr}(D^4,\partial,\tau)\simeq \left[S^4,\mathrm{SO}(4)\right]$$

and thus a fibration:

$$[S^4,\mathrm{SO}(4)]\longrightarrow \widetilde{\mathrm{B}\,\mathrm{Diff}}(D^4,\partial,\tau)\longrightarrow \mathrm{B}\,\mathrm{Diff}(D^4,\partial)$$

which we use to compute the isomorphism:

$$0=\pi_2[S^4,\mathrm{SO}(4)]\longrightarrow \pi_2\widetilde{\mathrm{B}\,\mathrm{Diff}}(D^4,\partial,\tau)\xrightarrow{\simeq}\pi_2\,\mathrm{B}\,\mathrm{Diff}(D^4,\partial)\longrightarrow \pi_1[S^4,\mathrm{SO}(4)]=0$$

Hence we see that we will be able to forget framing and focus on $\pi_2\,\mathrm{B}\,\mathrm{Diff}(D^4,\partial)\simeq \pi_1\,\mathrm{Diff}(D^4,\partial)$.

4. Configuration spaces and their compactifications

Differential-geometric blowups

In this subsection we consider a manifold X with possible boundary and corners and its submanifold Y also with possible corners. We require that $\partial Y\subset\partial X$ and that Y is transverse to ∂X. See [Arn83, §2] for a more detailed treatment of the topic.

Definition 4.1 (Blowup)**.** For a compact manifold X and a submanifold $Y\subset X$ transversal to (or intersecting vacuously with) ∂X we define *the blowup of X along Y*

$$\mathcal{B}\ell(X,Y)=\text{closure of }\Big(X\smallsetminus\mathrm{Int}\,\nu Y\smallsetminus\mathrm{Int}\,\nu Y|_{\partial Y}\Big)\text{ in }X$$

In other words we replace Y with its normal sphere bundle.

Example 4.2 (Blowup along a point)**.** Let $X=\mathbb{R}^2$ and $Y=(0,0)$. $\mathcal{B}\ell(X,Y)$ is then $S^1\times\mathbb{R}_+$. This procedure is equivalent to the polar substitution:

$$\begin{cases}x=r\sin\varphi\\ y=r\cos\varphi\end{cases}\quad\Longleftrightarrow\quad\begin{aligned}\mathbb{R}_+\times S^1&\longrightarrow\mathbb{R}^2\smallsetminus(0,0)\\(r,\varphi)&\longmapsto(x,y)\end{aligned}$$

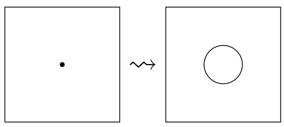


Figure 3: $[-1,1]\times[-1,1]$ blown up along $(0,0)$.

Example 4.3 (Blowup with boundary)**.** This time let

$$X=D^3=\{(x,y,z)\in\mathbb{R}^3|x^2+y^2+z^2\leq 1\}$$

and $Y = I = \{x = y = 0\}$. Blowing up X along Y is replacing Y with a tube. Note the behavior on ∂X i.e., we see here the necessity of $\smallsetminus \operatorname{Int} \nu Y|_{\partial Y}$ in the definition, and how blowups produce corners.

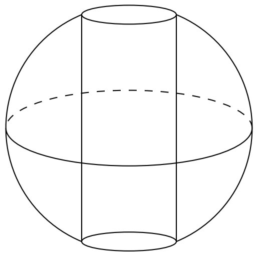


Figure 4: D^3 blown up along the vertical axis.

Lemma 4.4. $\mathcal{B}\ell(X, Y)$ is a manifold with corners[♠]. Moreover it admits the inclusion

$$\iota \colon \mathcal{B}\ell(X, Y) \hookrightarrow X \smallsetminus Y$$

which is a homotopy equivalence. Sometimes we call ι the blowup by abuse of notation.[♡]

Remark 4.5. The normal bundle to Y is a subbundle of TX. Blowing up can be seen as replacing Y with directions into (or out of) Y, thus vectors from TX.

It will be useful to view blown up manifolds not as manifolds with corners but as *stratified manifolds.*

Definition 4.6 (cf. [Arn83], p. 230)**.** A stratified submanifold of a smooth manifold is a finite union of mutually disjoint smooth manifolds (strata) satisfying the following condition: the closure of every stratum consists of the stratum itself and a finite union of strata of smaller dimensions.

Since configuration spaces naturally embed into euclidean spaces this definition is enough for our purposes.

Example 4.7. Consider

$$\{(x, y, z) \in \mathbb{R}^3 \mid x = 0 \vee y = 0 \vee z = 0\}$$

This space has a natural stratified structure: 12 open plane-quarters of dimension two e.g. $\{x > 0, y > 0, z = 0\}$, six open half-lines of dimension one e.g $\{x > 0, y = z = 0\}$, and the point $(0, 0, 0)$ of dimension zero. See Figure 5. We see that stratified manifold is a form of manifold with singularities.

Example 4.8. Consider $\mathbb{R}^2$ blown up in $(0, 0)$ and then along (the image of) $\{x = 0\}$. The result has a canonical stratified structure. See Figure 6. There are six codimension one strata: two pieces of the circle around $(0, 0)$ and four codimension two points. In higher dimensions, a blowup along a point and then along a line intersecting it has less strata since S^0 is the only non-connected sphere.

[♠] See [Joy10] for a careful definition.
[♡] Usually in the literature the projection is the *blowup*, while the inclusion is the *blow-down*. However in this context the opposite is more convenient

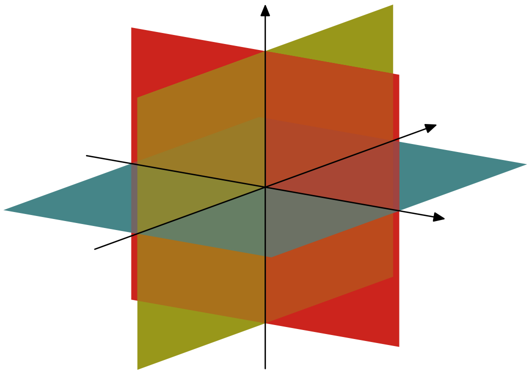


Figure 5: $\{(x,y,z) \in \mathbb{R}^3 \mid x = 0 \vee y = 0 \vee z = 0\}$

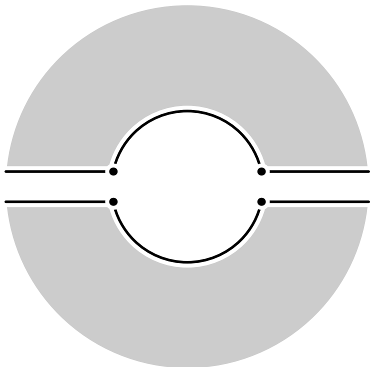


Figure 6: Stratified $\mathcal{B}\ell(\mathcal{B}\ell(\mathbb{R}^2, \mathbf{0}), \{x = 0\})$

Configuration spaces

Definition 4.9 (Configuration space)**.** Let X by any topological space. The space

$$C_n(X) \coloneqq \{(x_1, \dots, x_n) \in X^n \mid x_i = x_j \iff i = j\}$$

is called the configuration space of n points in X. One can also define $C_\Lambda(X)$ for $\Lambda \subset \{1, \dots, n\}$ by forgetting x_i for $i \notin \Lambda$. The space $C_\Lambda(X)$ is the naturally diffeomorphic to $C_{|\Lambda|}(X)$, but sometimes remembering which points we forgot is important.

The space $C_n(X)$ admits a tautological embedding

$$C_n(X) \hookrightarrow X^n$$

and natural forgetful maps

$$\psi_\Lambda \colon C_n(X) \longrightarrow C_\Lambda(X)$$

It is often convenient to think of $C_n(X)$ as $X^n \smallsetminus \bigcup_{\Lambda \in \{1,\dots,n\}} \Delta(\Lambda)$ where

$$\Delta(\Lambda) \coloneqq \{(x_1, \dots, x_n) \in X^n \mid x_i = x_j \iff i, j \in \Lambda\}$$

is called the Λ-diagonal or *sub-diagonal* when Λ is not specified.

One can think of C_n as a functor alternative to π_n instead of counting maps from spheres it counts maps from tuples of points. In other words, just as π_n is represented by S^n in the homotopy category, C_n is represented by n copies of a point, but in the category of topological spaces with only inclusions as morphisms. To make up for that fact – we want C_n to incorporate information about collisions of points. Just allowing them to collide it leaves us with $C_n(X) = X^n$ hence we need to remember more information: direction of the collision. Each collision corresponds to some sub-diagonal so we attach boundary to $C_n(X)$ along these sub-diagonals, finally constructing the compactification $\overline{C}_n(X)$. This attaching of boundary is best formalized by blow-ups.

We introduce examples before stating the technical definition. We start with collisions of two points. They look like limits of two points in X coming closer to each other, or equivalently, like a single point in X^2 approaching the diagonal. Hence $\overline{C}_2(X)$ should be X^2 blown-up along Δ. Provided it exists, a trivialization of the unit normal bundle to Δ is exactly the identification of $\partial\mathcal{B}\ell(X^2, \Delta)$ with $\Delta \times S^d$ hence it gives us a notion of direction of collision by point in S^d and the place of collision by point in Δ. See figure below.

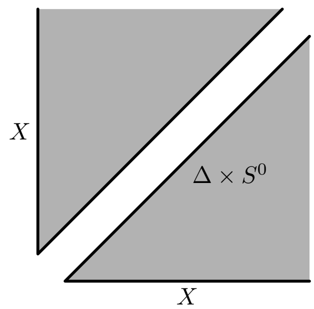


Figure 7: $\overline{C}_2(X)$. Since X is 1-dimensional on the drawing the boundary is $\Delta \times S^0$.

It is also worth noting that, provided both are trivial, the normal bundle to the diagonal is isomorphic to the tangent bundle of the diagonal by: $(x, x) \longmapsto (-x, x)$ in each fiber. This is the case for X parallelizable. In order to be coherent with Section 3, we will usually use the term *framing* for the trivialization.

Collisions of three or more points are more subtle. One way to see the reason behind is by observing that n-point sub-diagonals are contained in multiple $(n-1)$-point sub-diagonals (are even the their intersections). For the sake of simplicity let us examine $C_3(\mathbb{R})$. Assume we have already accounted for 2-point collisions i.e., blown up along $\Delta(1, 2)$, $\Delta(2, 3)$, $\Delta(1, 3)$ – replaced these planes with their double copies $\mathbb{R}^2 \times S^0$. For now we ignore any issues with the order of blow-ups. The big diagonal $\Delta(1, 2, 3)$ has already been blown up three times hence there are $2^3 = 8$ copies of it. A cross sections looks like in the figure below.

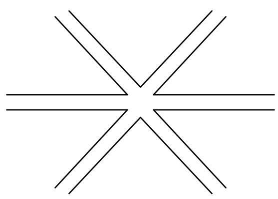


Figure 8

Each corner is a preimage of $\Delta(1, 2, 3)$ in the blow-down map to X^3. To model a triple collision we can approach $\Delta(1, 2, 3)$ along the boundary – by first colliding two points or through the interior only – performing a simultaneous collision. We want to distinguish these types of collisions. Technical way to do this is by blowing up $\Delta(1, 2, 3)$ first and only then the sub-diagonals. Conveniently the sub-diagonals become disjoint as their intersection had already been blown-up thus there are no problems with order of blow-ups. A cross-section of the result looks like figure below.

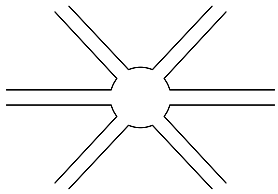


Figure 9

The "round" boundary corresponds to simultaneous collisions. From the other viewpoint – $C_3(\mathbb{R})$ as triples of points on a line – looks like in the figure below.

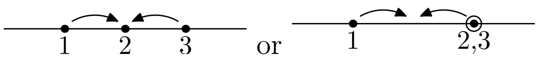


Figure 10

Observe that there are six "round" parts in Figure 9 corresponding to six possible arrangements of three points before the collision – six possible threefold-directions of collision.

It is also educational to analyze the $C_2(\mathbb{R}^2)$ since in this case the sub-diagonals no longer separate $(\mathbb{R}^2)^3$ and (equivalently) the sphere bundles have circles in fibers instead of S^0s allowing to see nontrivial behavior. The unit normal bundle to the big diagonal can be identified (by the framing!) with $S^3 \times \mathbb{R}^3$. In a cross-section, the direction of a threefold collision is parametrized by a point (or better called angle) in S^3. We want to see it as three points in $\mathbb{R}^2$. Say they meet in a distinguished point x (so we work on a cross-section over $(x, x) \in \Delta$). They each approach x from an angle $\varphi_i \in S^1$, $i \in \{1, 2, 3\}$. Three such angles parametrize the 3-sphere.

All of the above can be performed for arbitrary manifold with corners.

Definition 4.10 (Differential Fulton-MacPherson compactification)**.** Let X be a closed manifold with possible corners. Consider X^n and blow it up along $\Delta(1, \ldots, n)$, then along (the preimages in the blowdown of) $\Delta(\Lambda)$ for $|\Lambda| = n - 1$, then along $\Delta(\Lambda)$ for $|\Lambda| = n - 2$ and so on until $|\Lambda| = 2$. The resulting space is called $\overline{C}_n(X)$.

The following lemma is a straightforward corollary of definition of blowup.

Lemma 4.11. $\overline{C}_n(X)$ is a manifold with corners. Moreover the inclusion $C_n(X) \hookrightarrow \overline{C}_n(X)$ is a homotopy equivalence.

Remark 4.12. For a smooth map $X \longrightarrow Y$ it is obvious that the induced map $C_n(X) \longrightarrow C_n(Y)$ exists. This map extends to the boundary as the derivative by the identification in Remark 4.5. This phenomenon makes $\overline{C}_n$ an invariant tailored for distinguishing exotic structures, as it uses up one degree of differentiability.

As shown in [LX23] and [Che24], $\overline{C}_n$ is far from a perfect invariant, as it depends on slightly less than the differential structure – a formal smooth structure. That means it cannot directly distinguish between two exotic copies of the same manifold. However, it still detects exotic phenomena, example being the Watanabe's bundle.

Compactification of $C_*(\mathbb{R}^4)$

We also describe a compactification of $C_n(\mathbb{R}^4)$ – of a noncompact manifold. The ideological reason to do so is we are interested in $\mathrm{Diff}(D^4, \partial)$, hence we want to embed the points in $\mathrm{Int}\, D^4 \simeq \mathbb{R}^4$. For the compactification we need to attach boundary corresponding to collisions and to points going infinitely far from 0. In other words, the latter means choice of compactification of $\mathbb{R}^d$, which for us is the usual S^d.

For this procedure we need yet another way of viewing $C_n(X)$. It is a choice of the first point in X, then a choice of the second point in $X \smallsetminus *$, then of the third in $X \smallsetminus \{**\}$, and so on… One is tempted to claim that

$$C_n(X) \simeq X \times (X \smallsetminus *) \times (X \smallsetminus \{**\}) \times \dots$$

It is not true in general, but true locally: $C_n(X)$ is a fiber bundle over each of these components, in particular over X itself. Let ρ_n be the projection map $C_n(X) \longrightarrow X$. It obviously extends to $\overline{C}_n(X)$ smoothly. The fiber over any point in $x \in X$ is the configuration space of $n-1$ in $X \smallsetminus x$.

Definition 4.13. View S^4 as $\mathbb{R}^4 \cup \infty$. Let $C_n(S^4;\infty)$ denote the preimage of ∞ under $\rho_{n+1}\colon C_{n+1}(S^4) \longrightarrow S^4$. Since it is a subspace of a compact space $\overline{C}_{n+1}(S^4)$ it can be closed in it resulting in the compactification $\overline{C}_n(S^d;\infty)$.

Proposition 4.14. $\overline{C}_2(S^4;\infty)$ is homotopy equivalent to S^3. Thus

$$H^*(\overline{C}_2(S^4;\infty),\mathbb{Z}) \simeq \begin{cases} \mathbb{Z} & \text{for } * = 0 \text{ or } 3 \\ 0 & \text{otherwise} \end{cases}$$

Moreover

$$H^*(\overline{C}_2(S^4;\infty),\partial,\mathbb{Z}) \simeq \begin{cases} \mathbb{Z} & \text{for } * = 5 \text{ or } 8 \\ 0 & \text{otherwise} \end{cases}$$

Proof.

$$\overline{C}_2(S^4;\infty) = \text{closure of } C_2(\mathbb{R}^4) \simeq_{hty} C_2(\mathbb{R}^4) = \mathbb{R}^4\text{-bundle over } (\mathbb{R}^4 \smallsetminus *) \simeq_{hty} S^3$$

The second statement follows from the Poincaré–Lefschetz duality:

$$H^*(\overline{C}_2(S^4;\infty),\partial,\mathbb{Z}) \simeq H_{8-*}(\overline{C}_2(S^4;\infty),\mathbb{Z}) \simeq H_{8-*}(S^4,\mathbb{Z})$$

⊠

Codimension one strata

Observe that the construction of $\overline{C}_n(X;*)$ is well defined not only for spheres, but for any closed manifold X. In this subsection we give a description of the boundary of $\overline{C}_n(X;*)$. For X almost-parallelizable the result is particularly well-behaved. We analyze the codimension one strata, which form the full-measure part of boundary. Each such stratum corresponds to a subset of $N \cup *$ – collision of points that has been blown up.

Definition 4.15. Let $\mathring{X}$ denote the manifold X with a point cut out, choice of which is irrelevant. A manifold X^d is *almost-parallelizable* if $T\mathring{X}$ is a trivial $\mathbb{R}^d$-bundle over $\mathring{X}$.

Remark 4.16. Some authors use the term *almost-parallelizable* for manifolds whose tangent bundle becomes trivial after summing it with a trivial bundle of some dimension. These are different notions.

Note that all spheres are almost parallelizable, moreover for $d \neq 1, 3, 7$, S^d is not parallelizable itself, but almost parallelizable.

Until the end of this subsection X is an almost-parallelizable closed manifold. The boundary of $C_n(X; *)$ consists of various codimension 1 strata that correspond to collisions of certain points and their closures. Here we describe the strata explicitly.

Definition 4.17. Let V be an $\mathbb{R}$-linear space of dimension d. Define

$$\begin{aligned} C_n^*(V) &:= {C_n(V)}\Big/{\text{diagonal translations \& scaling}} \\ &= {V^n}\Big/{\substack{(x_1, \dots, x_d) \sim (x_1 + v_1, \dots, x_d + v_d) \\ (x_1, \dots, x_d) \sim (cx_1, \dots, cx_d)}} \qquad \text{for } v \in V^n (\simeq \mathbb{R}^{nd}),\ c \in \mathbb{R} \setminus 0 \end{aligned} \tag{$*$}$$

Similarity we define $C_\Lambda^*(V)$.

Remark 4.18. $C_n^*(V)$ can be also defined as a subspace of $C_n(V)$ cut out by equations:

$$\sum_{i=1}^{n} |x_i|^2 = 1, \quad \sum_{i=1}^{n} x_i = 0 \tag{$*^1$}$$

or alternatively:

$$\sum_{i=1}^{n-1} |x_i|^2 = 1, \quad x_n = 0 \tag{$*^2$}$$

Proof.

$(*^1) \rightsquigarrow (*^2)$: shift by $-x_n$, rescale as needed.

$(*^2) \rightsquigarrow (*^1)$: shift by $\frac{1}{n} \sum x_i$, rescale as needed.

any $\rightsquigarrow (*^1)$: shift by $-x_n$, rescale as needed. What is left is to prove that no elements in $(*^2)$ are identified by $(*)$. It is true, because any operation $(*^2) \rightsquigarrow (*^1)$ is diagonal and id on $x_n = 0$, hence is at most a scaling, but the other norms are fixed, so it is *id*. $\boxtimes$

Example 4.19. For two points the construction is quite simple.

$$C_2^*(V) = \left\{(x_1, x_2) \in V \times V \mid |x_1|^2 = 1,\ x_2 = 0\right\} = \text{the unit sphere in } V$$

We will use this isomorphism several times.

The space $C_n^*(V)$ admits two canonical constructions:

Definition 4.20. Define $\overline{C}_r^*(V)$ as the closure of $C_n^*(V)$ in $\overline{C}_n(V)$.

Definition 4.21. For a vector bundle $\pi \colon E \longrightarrow B$ define $C_n^*(\pi)$ by applying $C_n^*(-)$ fiberwise. This of course admits closure too, i.e. $\overline{C}_n^*(\pi)$.

The space $C_n^*(V)$ models a configuration behavior: it corresponds to relative configurations – we forget where the points actually are and only care about their distances to one another. This is useful when describing codimension one strata.

Suppose we have an almost-parallelizable manifold X and want to understand the codimension one strata of $\overline{C}_n(X)$. They occur as the point collide. Suppose a collision of subset Λ happens at point $x \in X$, that is we consider the subset of $\overline{C}_n(X)$ consisting of limits of configurations where $x_i \longrightarrow x$, $i \in \Lambda$. Denote this stratum S_Λ. The points x_i after the blow-up are represented by the directions in which they approach x hence they live in the tangent space T_xX. Their exact position in this space is not uniquely-defined, we may shift and scale them arbitrarily. It is then convenient to represent them 'centered around 0' like in $(*^1)$ or $(*^2)$. We care, however about their relative distances to one another as it may happen that some of them collide before the common collision, hence the actions of scaling and shifting have to be diagonal. Thus we have obtained $C^*_\Lambda(T_xX)$ with a canonical projection to $x \in X$. In fact the choice of x is redundant and we actually end up with $C^*_\Lambda(TX)$ – the 'configurated' and 'asterisked' $T\mathring{X}$ whose fiber is $C^*_\Lambda(\mathbb{R}^d)$. Finally, to account for points not from Λ we need to consider the space

$$C_{n,\Lambda}(X) \coloneqq \{(x_1, \dots, x_n) \in X^n \mid x_i = x_j \iff i = j \vee i, j \in \Lambda\}$$

isomorphic to $C_{n-|\Lambda|+1}(X)$. It parametrizes configurations of points not from Λ together with the point x, so can be thought of as $C_{N\smallsetminus\Lambda\cup x}(X)$. It is also equipped with a canonical map:

$$\vartheta\colon C_{n,\Lambda}(X) \longrightarrow X$$

that takes the Λ–collision points to their location in X, i.e., forgets the factors outside of Λ and maps the rest by projection to X. S_Λ is then the pullback of the $C^*_\Lambda(TX)$ bundle from single X to $C_{n,\Lambda}(X)$.

To take advantage of the almost-parallelizability of X we need to consider the space $C_n(X;x) \simeq C_n(\mathring{X})$. In the case $x \notin \Lambda$ the above description stays the same except for the fact that now S_Λ is a bundle over $C_{n,\Lambda}(\mathring{X})$ that comes from $T\mathring{X}$ by fiber substitution and pullback, thus is a trivial. Diagrammatically:

$$\begin{tikzcd}
S_\Lambda \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & C^*_{|\Lambda|}(T\mathring{X}) \arrow[d] \\
C_{n,\Lambda}(\mathring{X}) \arrow[r, "\mathring{\vartheta}"] & \mathring{X}
\end{tikzcd}$$

Write $N = \{1, \dots, n\}$. The compactification is straightforward:

Proposition 4.22. For an almost parallelizable closed manifold X^d and a subset $\Lambda \subset N$, $x \notin \Lambda$:

$$\overline{S}_\Lambda \simeq \overline{C}_{N\smallsetminus\Lambda+1}(X) \times \overline{C}^*_\Lambda(\mathbb{R}^d)$$

Remark 4.23. If X is not almost-parallelizable then we have a fiber bundle:

$$\overline{C}^*_\Lambda(\mathbb{R}^d) \longrightarrow \overline{S}_\Lambda \longrightarrow \overline{C}_{N\smallsetminus\Lambda+1}(X)$$

whose bundle isomorphism type depends only on $T\mathring{X}$.

It is, however, not straightforward how the component $\overline{C}^*_\Lambda(\mathbb{R}^d)$ looks like. First, observe that if $|\Lambda| = 2$ then by $(*^2)$, $C^*_\Lambda(\mathbb{R}^d) \simeq S^{d-1}$ so is already compact. For $|\Lambda| > 2$, what we get in the compactifications are all the sub-collisions: configurations of points from Λ with some subset colliding before the others. Hence, in $\overline{C}^*_\Lambda(\mathbb{R}^d)$ we have C^*_K for all subsets $K \subset \Lambda$. Inductively, the same phenomenon occurs for subsets with $|K| > 2$.

Example 4.24. As an example we include Figures 11 and 12 of points in compactifications of $C_6(S^2;\infty)$. Each figure is one point in $\overline{C}_6(S^2;\infty)$ – one configuration. We use funnels to depict closeness. When a point x is in the funnel over point y it means 'x is hits y' or 'x is infinitely close to y compared to all other points'.

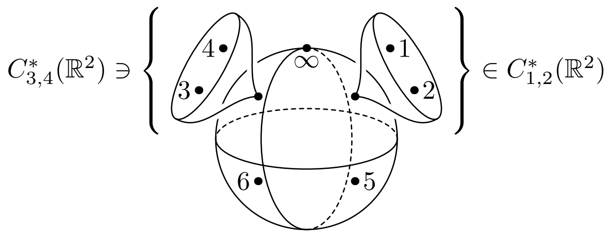


Figure 11: 1 hits 2 and 3 hits 4 not in ∞. In fact the points can never meet in ∞, only arbitrarily close to it. We abuse phrasing for simplicity.

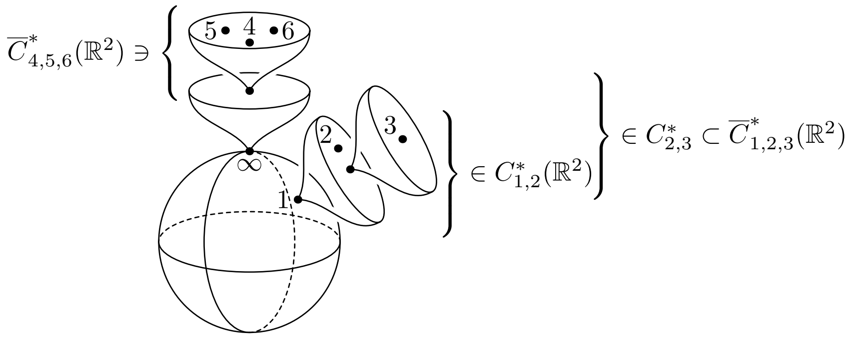


Figure 12: 2 hits 1 and then 3 hits them while 4, 5, 6 collide simultaneously in ∞. Note, that although 1, 2, 3 are collided, 1 is 'infinitely closer' to 2, than 3 is. Also note, that 4, 5, 6 need two funnels. It is because collision never happens in ∞, only 'infinitely close'.

In case $x \in \Lambda$ the reasoning is the same, but now the point of collision is fixed to $*$ therefore we pullback by the projection map:

$$\vartheta\colon C_{N\smallsetminus\Lambda}(\mathring{X}) \times x \longrightarrow x$$

and the bundle projected is:

$$\begin{array}{c} C^*_\Lambda(T_xX) \\ \downarrow \\ x \end{array}$$

The diagram looks as follows:

$$\begin{tikzcd} S_\Lambda \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & C^*_\Lambda(T_xX) \arrow[d] \\ C_{N\smallsetminus\Lambda}(\mathring{X}) \times x \arrow[r, "\vartheta"] & x \end{tikzcd}$$

Moreover observe that $|N \smallsetminus \Lambda| = n - (|\Lambda| - 1)$ because one of the point from Λ is not in N. Accounting for closures we end up with the proposition:

Proposition 4.25. For a closed manifold X^d and a subset $\Lambda \subset N \cup x$, $x \in \Lambda$:

$$\overline{S}_\Lambda \simeq \overline{C}_{N \smallsetminus \Lambda}(\mathring{X}) \times \overline{C}^*_\Lambda(\mathbb{R}^d)$$

Remark 4.26. Observe that for X almost-parallelizable, despite the differences in construction, in both cases we end up with isomorphic spaces:

$$\overline{S}_\Lambda = \overline{C}_{n-|\Lambda|+1}(\mathring{X}) \times \overline{C}^*_{|\Lambda|}(\mathbb{R}^d)$$

Example 4.27. As an example we describe the space $\overline{C}_2(S^4;\infty)$ in detail. It is the space of configurations of 2 points in S^4 with a distinguished point ∞ 'forbidden'. Its boundary is parametrized by the 4 subsets of $\{1,2,\infty\}$ of cardinality at least 2:

1. $S_{1,2} = \{$collisions of 1 and 2 but not in $\infty\}$
2. $S_{1,\infty} = \{$1 hitting ∞, but not 2$\}$
3. $S_{2,\infty} = \{$2 hitting ∞, but not 1$\}$
4. $S_{1,2,\infty} = \{$collision of 1 and 2 in $\infty\}$

We first blowup along $S_{1,2,\infty}$. Then along all other subsets. A schematic picture is similar to Figure 9.

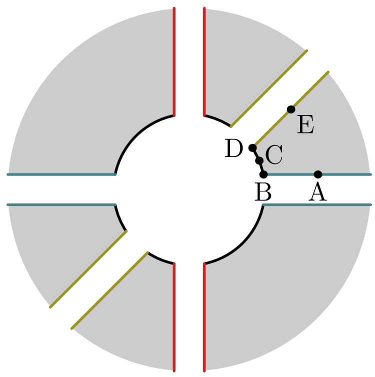


Figure 13: $\overline{C}_2(S^4;\infty)$. The four codimension one strata are denoted with different colors: $S_{1,2}$ with green, $S_{1,\infty}$ with blue, $S_{2,\infty}$ with red and $S_{1,2,\infty}$ with black. Their intersections, the corners, form the codimension two strata. The letters denote the five possible types of regions in the boundary. They are drawn in Figure 14

The codimension one strata can be explicitly described:

1. $\overline{S}_{1,2} = \overline{C}_1(\mathring{S}^4) \times \overline{C}^*_2(\mathbb{R}^4) \simeq \mathbb{R}^4 \times S^3$
2. $\overline{S}_{1,\infty} = \overline{C}_{\{2\}}(\mathring{S}^4) \times \overline{C}^*_2(T_\infty S^4) \simeq \mathbb{R}^4 \times S^3$
3. $\overline{S}_{2,\infty} = \overline{C}_{\{1\}}(\mathring{S}^4) \times \overline{C}^*_2(T_\infty S^4) \simeq \mathbb{R}^4 \times S^3$
4. $\overline{S}_{1,2,\infty} = \infty \times \overline{C}^*_3(T_\infty S^4)$

The isomorphisms above are diffeomorphisms, not only homotopy equivalences.

In Figure 13 we see there are five types of boundary regions, labeled A, B, C, D and E. We include figures from a tuples-on-manifold instead of product-without-diagonals point of view. See *Figure* 14

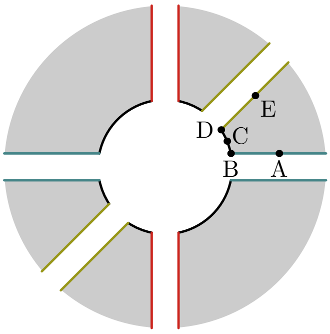
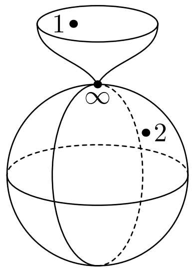
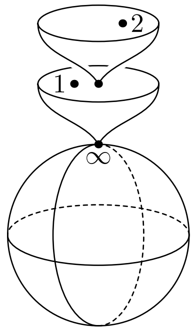


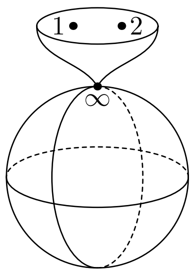
Figure 13 repeated for the Reader's convenience



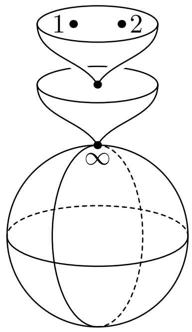
A. $S_{1,\infty}$



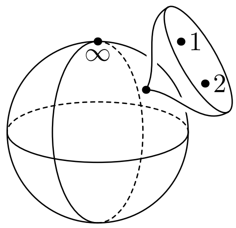
B. $S_{1,\infty} \cap S_{1,2,\infty}$



C. $S_{1,2,\infty}$



D. $S_{1,2} \cap S_{1,2,\infty}$



E. $S_{1,2}$

Figure 14: The five types of boundary regions of $\overline{C}_2(S^4; \infty)$. The S^4 is depicted as S^2.

5. Propagators

It is desirable to know the (co)homology of configuration spaces. Roughly speaking we expect any homology class of $C_n X$ to come either from X (due to product of Xs in the definition) or from collisions of points (the cut-out sub-diagonals). In the case $X = \mathbb{R}^d$ this heuristic happens to produce useful results: we are able to find some homology classes parametrized by collisions of points, moreover just two points. Propagators are exactly these classes. Due to Poincaré–Lefschetz duality we can think of them as differential forms. Here we describe the construction of propagators for $C_n(\mathbb{R}^4)$.

Proposition 5.1. The cohomology group $H^3_{\mathrm{dR}}(S^3)$ is generated by the volume one form:

$$\mathrm{vol}_{S^3} = \frac{1}{2\pi^2}\left(x_1 \,\mathrm{d}x_2 \,\mathrm{d}x_3 \,\mathrm{d}x_4 - x_2 \,\mathrm{d}x_1 \,\mathrm{d}x_3 \,\mathrm{d}x_4 + x_3 \,\mathrm{d}x_1 \,\mathrm{d}x_2 \,\mathrm{d}x_4 - x_4 \,\mathrm{d}x_1 \,\mathrm{d}x_2 \,\mathrm{d}x_3\right)$$

Moreover, the sphere admits an involution:

$$\begin{aligned} \iota\colon S^3 &\longrightarrow S^3 \\ x &\longmapsto -x \end{aligned}$$

that is homotopic to id, thus induces id on cohomology, in particular $\iota^*(\mathrm{vol}_{S^3}) = \mathrm{vol}_{S^3}$

Lemma 5.2 (Gauss map)**.** A map given by

$$\begin{aligned} \varphi\colon C_2(\mathbb{R}^4) &\longrightarrow S^3 \\ (x_1, x_2) &\longmapsto \frac{x_2 - x_1}{||x_2 - x_1||} \end{aligned}$$

called the *Gauss map* extends to the compactification as

$$\overline{\varphi}\colon \overline{C}_2(S^4;\infty) \longrightarrow S^3$$

Moreover, both φ and $\overline{\varphi}$ are homotopy equivalences.

Proof. We see that $C_2(\mathbb{R}^4) \simeq \mathbb{R}^4 \times (S^3 \times \mathbb{R})$. The map φ is the choice of direction x_1 to x_2. It factors through the projection to $S^3 \times \mathbb{R}$ since it is invariant to diagonal shifts of x_1 and x_2, and acts as projection to S^3. Both of these reductions are homotopy equivalences.

Continuous extension to the boundary is straightforward: φ returns the angle of collision. It is smoothness that requires a careful proof, see [Wat23, Lemma C.11]. ⊠

Note that the Gauss map is coordinate-dependent. We now construct its analogue that requires framing instead. This will be useful for a bundle of configuration spaces, where he we have no global coordinates, but framing defined in Section 3 provides necessary structure for trivialization of tangent bundle over each fiber.

Pick a standard at ∞ framing[♣] $\tau\colon T(S^4 \setminus \infty) \xrightarrow{\simeq} (S^4 \setminus \infty) \times \mathbb{R}^4$. By Remark 4.5 it induces a map $p(\tau)\colon \partial\overline{C}_n(S^4;\infty) \longrightarrow S^3$. On the codimension one strata $p(\tau)$ can be explicitly written as:

$$p(\tau) = \begin{cases} S_{1,2} = C_1(\mathring{S}^4) \times C_2^*(\mathbb{R}^4) \xrightarrow{\tau} \mathbb{R}^4 \times S^3 \xrightarrow{\text{projection}} S^3 \\ S_{1,\infty} = C_{\{2\}}(\mathring{S}^4) \times C_2^*(T_\infty S^4) \xrightarrow{\text{projection}} C_2^*(T_\infty S^4) \simeq S^3 \\ S_{2,\infty} = C_{\{1\}}(\mathring{S}^4) \times C_2^*(T_\infty S^4) \xrightarrow{\text{projection}} C_2^*(T_\infty S^4) \simeq S^3 \\ S_{1,2,\infty} = \infty \times C_3^*(T_\infty S^4) \xhookrightarrow{(*^2)} C_2(\mathbb{R}^4 \setminus 0) \xrightarrow{\varphi} S^3 \end{cases}$$

We see that $p(\tau)$ agrees with $\overline{\varphi}$ as both maps return the angle between two points.

Lemma 5.3. The 3-form $p(\tau)^*\mathrm{vol}_{S^3}$ extends smoothly to a 3-form ω on $\overline{C}_n(S^4;\infty)$. Moreover, such extension is unique in cohomology.

Proof. See Lemma 5.5. ⊠

Definition 5.4. The cohomology class $[\omega]$ from the above lemma is called *propagator in fiber.*

Suppose we are given a (D^4,∂)-bundle $\pi\colon E \longrightarrow B$, where B is a compact manifold with possible boundary, equipped with a vertical framing τ_E. We transform it to a (S^4,∞)-bundle by collapsing the boundary of each fiber to a point. Then we perform a fiberwise configuration: replace each fiber with $\overline{C}_n(S^4;\infty)$. It is possible since configuration spaces admit the diagonal $\mathrm{Diff}(S^4;\infty)$ action. Denote this bundle

$$\overline{C}_n(\pi)\colon E\overline{C}_n(\pi) \longrightarrow B$$

For $n = 2$ we define a propagator for this bundle.

The framing τ_E induces a map

$$p(\tau_E)\colon \partial^v E\overline{C}_n(\pi) \longrightarrow S^3$$

Recall, that by definition it is required to be standard at ∞ in all fibers.

[♣] See the passage below Definition 3.3 for the meaning of *standard*.

Lemma 5.5. The 3-form $p(\tau_E)^*\mathrm{vol}_{S^3}$ extends smoothly to a 3-form ω on $\partial^v E\overline{C}_2(S^4;\infty)$. Moreover, such extension is unique in cohomology.

Proof. Consider the Leray-Serre spectral sequence for the relative fibration:

$$(\overline{C}_2(S^4;\infty),\partial) \longrightarrow (E\overline{C}_2(S^4;\infty),\partial^v) \longrightarrow B$$

The E_2 page is given by the formula

$$\begin{aligned} E_2^{p,q} = H^p(B;H^q(\overline{C}_n(S^4;\infty),\partial)) &= H^p(B)\otimes H^q(\overline{C}_2(S^4;\infty),\partial) \\ &= H^p(B)\otimes 0 \quad \text{for } q\neq 5,8 \quad \text{(Proposition 4.14)} \end{aligned}$$

therefore the sequence for $q<5$ converges to

$$H^n(E\overline{C}_2(S^4;\infty),\partial^v)=0 \quad \text{for } n\neq 5,8$$

Now consider the relative exact sequence of the pair $(E\overline{C}_2(\pi),\partial^v)$, we see the isomorphism:

$$0=H^3(E\overline{C}_2(\pi),\partial^v)\to H^3(E\overline{C}_n(\pi)) \xrightarrow{\simeq} H^3(\partial^v E\overline{C}_2(\pi)) \to H^4(E\overline{C}_2(\pi),\partial^v)=0$$

Hence any closed form of the vertical boundary extends to the whole space $E\overline{C}_2(\pi)$

For uniqueness consider any other extension ω'. The difference $\omega-\omega'$ vanishes on the boundary thus is in the image of the natural arrow

$$H^3(E\overline{C}_2(\pi),\partial^v) \longrightarrow H^3(E\overline{C}_2(\pi))$$

which is zero. ⊠

Definition 5.6. The cohomology class $[\omega]$ from the above lemma is called *propagator for framing τ_E*.

Proposition 5.7. Propagators extend overs cobordisms. That is: for a framed (D^4,∂)–bundle $(\pi\colon E\longrightarrow B,\tau_E)$ over a cobordism B between manifolds B_1 and B_2 consider propagators ω_1 and ω_2 for τ_E on $E\overline{C}_2(\pi)$ suitably restricted. Then there exists a propagator ω for τ_E on the whole $E\overline{C}_2(\pi)$ that extends both ω_1 and ω_2.

Proof. We pullback ω_1 and ω_2 to the collar neighborhoods $B_i\times[-\varepsilon,\varepsilon]$, without changing notation. The identification of the collar neighborhoods with $B_i\times[-\varepsilon,\varepsilon]$ can be chosen such that it is compatible with τ_E. Define

$$B' = B \smallsetminus \mathrm{B}_1\times[0,\varepsilon] \smallsetminus B_2\times[0,\varepsilon]$$

By Lemma 5.5 there exists a propagator for τ_E on $E\overline{C}_2\pi$, call it ω_a. We now have two propagators on each collar neighborhood, hence, again by Lemma 5.5, they differ by exact forms that vanish on the vertical boundary. In other words, there exist μ_1 and μ_2 in such that

$$\omega_a-\omega_1=\mathrm{d}\mu_1, \quad \omega_a-\omega_2=\mathrm{d}\mu_2$$

on $B_i\times[-\varepsilon,\varepsilon]$. The 2-forms μ_i can be extended to the form μ on the whole space $E\overline{C}_2(\pi)$ that also vanishes on the vertical boundary, since they are defined on disjoint closed codimension zero subsets. Consider a smooth function $\chi\colon E\overline{C}_2(\pi)\longrightarrow[0,1]$ that takes value 1 on $\overline{C}_2(\pi)^{-1}(\partial B)$ and 0 on $\overline{C}_2(\pi)^{-1}(B')$. Define the 3-form

$$\omega=\omega_a+\mathrm{d}(\chi\mu) \quad \text{on } E\overline{C}_2(\pi)$$

We see that

$$\begin{aligned}\omega|_{\partial^v E\overline{C}_2(\pi)} &= \omega_a|_{\partial^v E\overline{C}_2(\pi)}\\ \omega|_{\overline{C}_2(\pi)^{-1}B_i} &= \omega_i\end{aligned}$$

So ω is a propagator for τ_E that extends both ω_1 and ω_2. $\boxtimes$

Remark 5.8. A propagator in a fiber is a class in $H^3_{\mathrm{dR}}(\overline{C}_2(S^4;\infty))$. By the Poincaré–Lefschetz duality this group is isomorphic to $H_5(\overline{C}_2(S^4;\infty), \partial\overline{C}_2(S^4;\infty), \mathbb{R})$, hence we can represent a propagator as a class in such. This approach is employed in [Wat19] and thoroughly described in [Les24]. In our case, the explicit image of the propagator is rather hard to compute, however due to Proposition 4.14 we need not do that, as both groups are isomorphic to the coefficient module. By definition, the chain representing the propagator needs to have boundary in $\partial\overline{C}_2(S^4;\infty)$. See Figure 15 for an example.

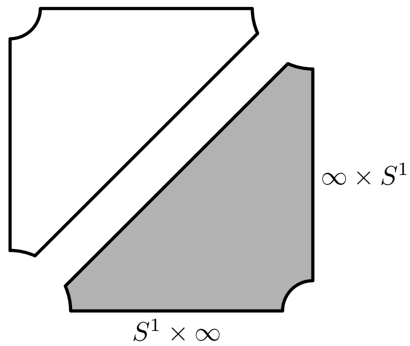


Figure 15: A propagator for S^1. The Gauss map is replaced by $S^1 \times S^1 \longrightarrow S^0$ with the same formula.

6. Configuration space integrals

In a single manifold

Recall graphs from Section 2. I to each graph we introduce a form that counts the collisions of vertices connected by edges. Choose a graph $\Gamma \in \mathcal{G}$ with v vertices, e edges and its embedding into $\operatorname{Int} D^4 \simeq \mathbb{R}^4$. Since it is a codimension three embedding, for a choice of $|v|$ points embedded in $\mathbb{R}^4$, there is only one way up to isotopy of embedding the edges. Thus every such embedding is associated to a configuration of v points in D^4 up to ordering of the vertices. Choose the ordering arbitrarily. Moreover, choose orientations of edges of Γ, also arbitrarily. We have a forgetful map

$$\psi_i \colon \overline{C}_v(\mathbb{R}^4) \longrightarrow \overline{C}_2(\mathbb{R}^4)$$

defined by for each edge by forgetting the vertices outside the edge i. Define a linear map

$$\begin{aligned}\omega_{\mathit{fib}} \colon \mathcal{G} &\longrightarrow \Omega^{3e}_{dR}(\overline{C}_v(S^4;\infty)) \quad [\diamond]\\ \Gamma &\longmapsto \bigwedge_{i\colon \text{edges of } \Gamma} \psi_i^*\overline{\varphi}^*\mathrm{vol}_{S^3} = \bigwedge_{i\colon \text{edges of } \Gamma} \psi_i^*\omega\end{aligned}$$

[$\diamond$] *fib* stands for *fiber* since we will later define this notion for families of manifolds

Recall that edges of Γ are labeled, thus the order of propagators in the wedge is uniquely determined. To every graph we assign a $3e$-form on an $4v$-dimensional manifold. We want to integrate it later, so we need $3e = 4v$, while keeping the graphs at least trivalent: $3v \leq 2e$ (for every vertex we need at least three edges and one edge can be used for two vertices). These lead to a contradiction: $4v = 3e \geq \frac{9}{2}v$. The remedy is bundleizing this construction. We will have a $3e$-form on a dim $= 4v$-bundle over a 2 dimensional manifold, that leads to $3e = 4v + 2$. This equation has a solution compatible with at-least-trivalency: $e = 6, v = 4$.

It is however worth noting, that in dimension 3, the equality we get is $2e = 3v$. Therefore the procedure produces invariants of 3-manifolds, not of bundles. See [Les24] or [LMO98].

In a family

Recall the bundle

$$\overline{C}_n(\pi)\colon E\overline{C}_n(\pi) \longrightarrow B$$

As previously, we have the forgetful maps in each fiber

$$\psi_i\colon \overline{C}_v(\mathbb{R}^4) \longrightarrow \overline{C}_2(\mathbb{R}^4)$$

which induce

$$E\psi_i\colon E\overline{C}_v(\pi) \longrightarrow E\overline{C}_2(\pi)$$

We finally have the linear map

$$\begin{aligned} \omega\colon \mathcal{G} &\longrightarrow \Omega^{3e}_{dR}(E\overline{C}_v(\pi)) \\ \Gamma &\longmapsto \bigwedge_{i\colon \text{edges of } \Gamma} (E\psi_i)^*\omega \end{aligned}$$

with $3e$-forms on an $8v + \dim B$-manifold in its image. Note that $\omega(\Gamma)$ is closed as a pullback of a closed form. Using the Fubini's Theorem we can integrate these forms along fibers. We obtain a linear map

$$\begin{aligned} I\colon \mathcal{G} &\longrightarrow \Omega^{(3e-4v)}_{dR}(B) \\ \Gamma &\longmapsto \overline{C}_v(\pi)_*\omega(\Gamma) \end{aligned}$$

We denote the process of integration $\overline{C}_v(\pi)_*$ since it is in fact a pushforward by the projection map.

When we take into account the grading on $\mathcal{G}$ by $k = e - v$, $l = 2e - 3v$ we see that I is in fact a map between two l-graded linear spaces:

$$I\colon P_k\mathcal{G}_l \longrightarrow \Omega^{(k+l)}_{dR}(B)$$

Theorem 6.1 (Kontsevich, Watanabe)**.**

(1) I is a chain map up to sign:

$$dI(\Gamma) = (-1)^{k+l+1}I(\delta\Gamma)$$

hence induces

$$I_*\colon P_kH^l(\mathcal{G};\mathcal{A}_k) \longrightarrow H^{k+l}(B;\mathcal{A}_k \otimes \mathbb{R})$$

(2) I_* is independent of the choice of the propagator

(3) I_* is independent of the choice of directions of edges

(4) I_* is invariant with respect to homotopy of τ_E

(5) I_* is natural with respect to bundle morphisms. That is: for a bundle morphism of framed bundles over base spaces B and B' induced by the map $f\colon B \longrightarrow B'$ the following diagram commutes:

$$\begin{tikzcd} P_k H^l(\mathcal{G}, \mathbb{Q}) \arrow[r, "I_*"] \arrow[dr, "I_*"'] & H^{k+l}(B, \mathbb{R}) \\ & H^{k+l}(B', \mathbb{R}) \arrow[u, "f^*"'] \end{tikzcd}$$

Hence I_* is sometimes called the *Kontsevich characteristic class.*

Remark 6.2. The whole Section 2 is motivated by Theorem 6.1(1). The graph differential and thus IHX are chosen as they are, in order to have a chain map. Anti-symmetry of graphs mirrors anti-symmetry of differential forms. Reason for trivalency can be found in the proof, see Lemma 7.4.

The gist of the theory is that the suitable differential on th graph space looks very natural thus provides a convenient combinatorial framework.

We could have introduced the whole Section 2 only now as a natural consequence of Theorem 6.1, however it would lead to some awkward arguments and notations.

Remark 6.3. In principle, the integral measures the degree of the Gauss map for each edge, this suggests that the result should be integral itself. It is however nit true, because of the compactification. We cannot even assume rationality of the integral.

7. Proof of Theorem 6.1

Before the proof we need to introduce a number of lemmas. We begin with a crucial general theorem.

Theorem 7.1 (Stokes theorem for fibre bundles, see [BT94] for reference)**.** For a fiber bundle $\pi\colon E \longrightarrow B$ with a compact oriented n-dimensional fiber, its restriction to fiberwise boundary $\pi^{\partial}\colon \partial^v E \longrightarrow B$ and a p-form α on E, where $n \leq p$:

$$d\pi_*\alpha = \pi_* d\alpha + (-1)^{p-n}\pi_*^{\partial}\alpha$$

We will now describe the form $\omega(\Gamma)$ and use Stokes theorem to compute the value of the integral. We replace $(S^4; \infty)$ with arbitrary almost-parallelizable, pointed, closed 4-manifold (X, x) to emphasize proof's independence of S^4. Nonetheless, the Theorem is well-stated only for S^4. The author is currently on a definition of propagator for other manifolds, which is the main missing ingredient.

Recall the description of codimension 1 strata of $\overline{C}_v(\pi)$ from Section 4. It carries to the bundleized version without changes that is:

$$\partial^v E\overline{C}_v(\pi) = \bigcup_{\Lambda \subset N \cup \infty} E\overline{S}_\Lambda$$

$$E\overline{S}_\Lambda \simeq E\overline{C}_{v,\Lambda}(\pi) \times \overline{C}^*_{|\Lambda|}(\mathbb{R}^4) \qquad \text{if } \infty \notin \Lambda$$

$$E\overline{S}_\Lambda \simeq E\overline{C}_{N\smallsetminus\Lambda}(\pi) \times \overline{C}^*_{|\Lambda|}(\mathbb{R}^4) \qquad \text{if } \infty \in \Lambda$$

Denote the projections p_1 and p_2. Let π^Λ denote the restriction of $E\overline{C}_v(\pi)$ to $E\overline{S}_\Lambda$. Recall also that $\overline{C}_{v,\Lambda}(X) \simeq \overline{C}_{N\smallsetminus\Lambda}(X) \simeq \overline{C}_{v-|\Lambda|+1}(X)$ which is also true in the bundleized version.

Provided Λ is a subset of N ($\infty \notin \Lambda$) denote with Γ_Λ the subgraph of Γ spanned by Λ and Γ/Λ the result of contracting Γ_Λ in Γ. These will correspond with the factors of $S_\Lambda \simeq C_{n,\Lambda} \times C^*_\Lambda(\mathbb{R}^4)$ which roughly parametrize 'configurations of not Λ' and 'configurations of Λ collided'.

Proposition 7.2. The form $\omega(\Gamma)$ restricted to strata decomposes as:

$$\begin{aligned} \omega(\Gamma)|_{E\overline{S}_\Lambda} &= \pm p_1^*\omega(\Gamma/\Lambda) \wedge p_2^*\omega(\Gamma_\Lambda) && \text{if } \infty \notin \Lambda \\ \omega(\Gamma)|_{E\overline{S}_\Lambda} &= \pm p_1^*\omega(\Gamma/(N \smallsetminus \Lambda)) \wedge p_2^*\omega(\Gamma_{N\smallsetminus\Lambda}) && \text{if } \infty \in \Lambda \end{aligned}$$

Where the sign is the sign of the permutation

$$\{1, \ldots, e\} \longrightarrow \{\text{edges of } \Gamma/\Lambda\} \wedge \{\text{edges of } \Gamma_\Lambda\}$$

Proof. Proof of both cases is the same. We present the proof of the first one.

We need to justify why these pullbacks make sense, since a priori the domains do not agree. Recall the definition of $\omega(\Gamma)$:

$$\bigwedge_{i\colon \text{edges of } \Gamma} \left(E\overline{C}_v(\pi) \xrightarrow{E\psi_i} E\overline{C}_{\text{endpoints of } i}(\pi) \right)^* \omega$$

It can be rewritten as:

$$\bigwedge_{i\colon \text{edges of } \Gamma/\Lambda} E\psi_i^*\omega \quad \wedge \quad \bigwedge_{i\colon \text{edges of } \Gamma_\Lambda} E\psi_i^*\omega$$

For the first term we observe that the forgetful map $E\psi_i$ factors through p_1 as:

$$\begin{tikzcd} E\overline{C}_v(\pi) \arrow[rr, "E\overline{\psi}_i"] \arrow[dr, "p_1"'] & & E\overline{C}_{\text{endpoints of } i}(\pi) \\ & E\overline{C}_{v,\Lambda}(\pi) \arrow[ur] & \end{tikzcd}$$

as we may first forget the points from Λ and then the rest.

The second term is more complicated. It is explained by the diagram:

$$\begin{tikzcd} \overline{S}_\Lambda \simeq \overline{C}_{\text{vertices of } \Gamma/\Lambda}(X;x) \times \overline{C}^*_{|\Lambda|}(\mathbb{R}^4) \arrow[r] \arrow[d, "p_2"'] & \overline{S}_{\text{endpoints of } i} \arrow[r, hook] \arrow[d, "\simeq"'] & \overline{C}_{\text{endpoints of } i}(X;x) \arrow[d, "\varphi"] \\ \overline{C}^*_{|\Lambda|}(\mathbb{R}^4) \arrow[ur] & \overline{C}_{2-2+1=1}(X;x) \times \overline{C}^*_2(\mathbb{R}^4) \arrow[r, "\varphi'"] \arrow[d, "\text{projection}"'] & S^3 \\ & C^*_2(\mathbb{R}^4) \arrow[ur, "\simeq"'] & \end{tikzcd}$$

We skip E and draw the diagram in a single fiber for readability. Note that when we work in a fiber, $\overline{C}_n(X)$ admits coordinates, thus φ is well defined. This time, the forgetful map $\overline{\psi}_i$ factors through the stratum corresponding the collision of the endpoints of i. The map φ' is the induced Gauss map on the subspace $\mathring{X} \times \overline{C}_2^*(\mathbb{R}^4) \subset \overline{C}_2(X)$ hence the square commutes. Moreover, since $\overline{C}_2^*(\mathbb{R}^4) \simeq S^3$ it already is the space of directions of collisions, φ' is a projection composed with id.

We remark writing $p_2^*\omega(\Gamma_\Lambda)$ is a slight abuse of notation.

The signs are self-explanatory. $\boxtimes$

Remark 7.3. The above description is also valid for when X is not almost-parallelizable, except for the fact that only locally. That is: let $\mathcal{U} = \bigcup U_i$ be the trivializing cover for the $\overline{C}_\Lambda^*(T\mathring{X})$-bundle over $E\overline{S}_\Lambda$ (induced from this bundle over $E\overline{C}_{n,\Lambda}(\pi)$). Over any U_i the projection to the fiber is now well defined, call it p_2^i. Then

$$\omega(\Gamma)|_{U_i} = \pm p_1^*\omega(\Gamma/\Lambda) \wedge p_2^{i*}\omega(\Gamma_\Lambda)$$

where p_1 is suitably restricted. For $\infty \in \Lambda$ this need not be done, the product structure is independent of X.

We now use this decomposition of $\omega(\Gamma)$ to apply Fubini's theorem. We show that most integrals along $\overline{C}_{|\Lambda|}^*(\mathbb{R}^d)$ vanish.

Lemma 7.4. When $|\Lambda| \geq 3$

$$\pi_*^\Lambda \omega(\Gamma) = 0$$

Proof. There are four cases:

1. Γ_Λ has all vertices of valence 3
2. Γ_Λ has a 2-valent vertex
3. Γ_Λ has a univalent vertex
4. Γ_Λ has a vertex with 0 edges

Case (1): We use a dimensional argument. Let v' and e' denote the numbers of vertices and edges of Γ_Λ. The trivalency condition implies

$$2e' - 3v' \geq 0$$

Due to the fiber bundle structure of $E\overline{S}_\Lambda$ we first integrate along the fiber $\overline{C}_{|\Lambda|}^*(\mathbb{R}^d)$, ($|\Lambda| = v'$), which is of dimension $4(v'-1)-1$. On the other hand $\omega(\Gamma_\Lambda)$ is e'-ary wedge of of 3-forms. For the integral to be non-zero we need $\deg \omega(\Gamma_\Lambda)$ to be precisely the dimension of the space on which we integrate, that is:

$$4(v'-1) - 1 = e'(4-1) \iff e' = \frac{4(v'-1)-1}{(4-1)}$$

Combining the above, we get the inequality

$$\begin{aligned} 2\frac{4(v'-1)-1}{3} &\geq 3v' \\ 8(v'-1) - 2 &\geq 9v' \\ 8v' - 10 &\geq 9v' \end{aligned}$$

which is impossible for $v' > 0$.

Case (2): Again, as we are integrating along a fiber we can choose coordinates. The Gauss map φ is then well defined, and agrees with $p(\tau_E)$ where it makes sense.

Call the 2-valent vertex a, and the two adjacent vertices b and c. We may assume they are different since multiple edges produce $\omega(\Gamma)=0$. Consider $C_\Lambda(\mathbb{R}^4)$ with position of point $\lambda \in \Lambda$ denoted $x_\lambda \in \mathbb{R}^4$. Recall that $C^*_\Lambda(\mathbb{R}^4)$ is a subset of $C_\Lambda(\mathbb{R}^4)$ given by the equation:

$$\sum_{i \in \Lambda \smallsetminus \kappa} |x_i|^2 = 1, \quad x_\kappa = 0$$

where κ is any element of Λ.

Consider the automorphism ι_Λ that acts on the a-th component as

$$x_a \longmapsto x_b + x_c - x_a =: \tilde{x}_a$$

and is id on all other components. It is only well defined on $C^*_\Lambda(\mathbb{R}^4) \smallsetminus C$, where C is the codimension 4 subset of $C_\Lambda(\mathbb{R}^4)$ given by $x_e = x_b + x_c - x_a$, for some $e \neq a$. The restriction of C to $C^*_\Lambda(\mathbb{R}^4)$ is also of codimension 4.

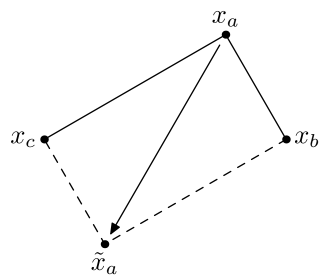


Figure 16: Note that the edges connecting x_b with x_a and $\tilde{x}_a$ both represent the edges of Γ with endpoints in a and b but in spaces $C^*(\mathbb{R}^4)$ and $\iota_\Lambda\big(C^*(\mathbb{R}^4)\big)$. Hence the map $\varphi'\psi_{ab}$ that takes direction between x_a i x_b is also the map that takes the direction between $\tilde{x}_a$ and x_b depending on where a is located. Similarly for x_c.

Recall ι of Proposition 5.1. We have the following commutative diagrams:

$$\begin{tikzcd}
C^*_\Lambda(\mathbb{R}^4) \arrow[r, "\varphi'\psi_{ac}"] \arrow[d, "\iota_\Lambda"'] & S^3 \arrow[d, "\iota"] \\
C^*_\Lambda(\mathbb{R}^4) \arrow[r, "\varphi'\psi_{ab}"'] & S^3
\end{tikzcd}
\qquad
\begin{tikzcd}
C^*_\Lambda(\mathbb{R}^4) \arrow[r, "\varphi'\psi_{ab}"] \arrow[d, "\iota_\Lambda"'] & S^3 \arrow[d, "\iota"] \\
C^*_\Lambda(\mathbb{R}^4) \arrow[r, "\varphi'\psi_{ac}"'] & S^3
\end{tikzcd}$$

$$\begin{tikzcd}
(\ldots, x_a, x_b, x_c, \ldots) \arrow[r, mapsto] \arrow[d, mapsto] & \frac{x_a - x_c}{|x_a - x_c|} \arrow[d, mapsto] \\
(\ldots, \tilde{x}_a, x_b, x_c, \ldots) \arrow[r, mapsto] & \frac{\tilde{x}_a - x_b}{|\tilde{x}_a - x_b|}
\end{tikzcd}
\qquad
\begin{tikzcd}
(\ldots, x_a, x_b, x_c, \ldots) \arrow[r, mapsto] \arrow[d, mapsto] & \frac{x_a - x_b}{|x_a - x_b|} \arrow[d, mapsto] \\
(\ldots, \tilde{x}_a, x_b, x_c, \ldots) \arrow[r, mapsto] & \frac{\tilde{x}_a - x_c}{|\tilde{x}_a - x_c|}
\end{tikzcd}$$

that induce a chain of equalities:

$$\begin{aligned}
\iota^*_\Lambda(\psi^*_{ab}\varphi'^*\mathrm{vol}_{S^3} \wedge \psi^*_{ac}\varphi'^*\mathrm{vol}_{S^3}) &= \iota^*_\Lambda\psi^*_{ab}\varphi'^*\mathrm{vol}_{S^3} \wedge \iota^*_\Lambda\psi^*_{ac}\varphi'^*\mathrm{vol}_{S^3} \\
&= \psi^*_{ac}\varphi'^*\mathrm{vol}_{S^3} \wedge \psi^*_{ab}\varphi'^*\mathrm{vol}_{S^3} \\
&= -\psi^*_{ab}\varphi'^*\mathrm{vol}_{S^3} \wedge \psi^*_{ac}\varphi'^*\mathrm{vol}_{S^3}
\end{aligned}$$

and ι_Λ acts trivially on other edge forms. Thus $\iota^*_\Lambda\omega(\Gamma_\Lambda) = -\omega(\Gamma_\Lambda)$. Moreover ι_Λ does not change the orientation since its is a combination of a shift by $x_b + x_c$ and

$x_a \longmapsto -x_a$ which is isotopic to identity on $\mathbb{R}^4$. We know that $\omega(\Gamma_\Lambda)$ is integrable over $\overline{C}^*_\Lambda(\mathbb{R}^4)$ so we can remove codimension > 0 subsets C and ∂ and still have the same value of the integral. We see that:

$$\int_{C^*_\Lambda(\mathbb{R}^4)\smallsetminus C} \omega(\Gamma_\Lambda) = \int_{\iota_\Lambda\left(C^*_\Lambda(\mathbb{R}^4)\smallsetminus C\right)} \omega(\Gamma_\Lambda) = \int_{C^*_\Lambda(\mathbb{R}^4)\smallsetminus C} \iota^*_\Lambda \omega(\Gamma_\Lambda) = \int_{C^*_\Lambda(\mathbb{R}^4)\smallsetminus C} -\omega(\Gamma_\Lambda)$$

where domains are of equal orientation. Thus

$$\int_{C^*_\Lambda(\mathbb{R}^4)\smallsetminus C} \omega(\Gamma_\Lambda) = 0$$

Case (3): Call the univalent vertex of Γ_Λ a, and its only neighbor b. The maps $\varphi'\psi_{ab}$ and $\varphi'\psi_{\alpha\beta}$ for $\alpha, \beta \in \Lambda$ and $\alpha \neq a$ factor as follows:

$$\begin{tikzcd} C^*_\Lambda(\mathbb{R}^4) \arrow[rr, "\varphi'\psi_{ab}"] \arrow[dr, "q"'] & & S^3 \\ & C^*_{\Lambda\smallsetminus a}(\mathbb{R}^4) \times S^3 \arrow[ur, "\text{projection}"'] & \end{tikzcd} \qquad \begin{tikzcd} C^*_\Lambda(\mathbb{R}^4) \arrow[rr, "\varphi'\psi_{\alpha,\beta}"] \arrow[dr, "q"'] & & S^3 \\ & C^*_{\Lambda\smallsetminus a}(\mathbb{R}^4) \times S^3 \arrow[ur, "\varphi'\psi_{\alpha,\beta}"'] & \end{tikzcd}$$

where

$$q(x_1, \ldots, x_{|\Lambda|}) = \left(\mu x_1, \ldots \widehat{\mu x_a}, \ldots, \mu x_{|\Lambda|}, \frac{x_a - x_b}{|x_a - x_b|}\right) \quad \text{and} \quad \mu = \frac{1}{\sqrt{1 - |x_a|^2}}$$

Therefore the form $\omega(\Gamma_\Lambda)$ on $C^*_\Lambda(\mathbb{R}^4)$ is a pullback by q of a $3e'$-form on $\overline{C}^*_{\Lambda\smallsetminus a}(\mathbb{R}^4) \times S^3$ and

$$3e' = \dim C^*_\Lambda(\mathbb{R}^4) = 4(|\Lambda| - 1) - 1 > \dim C^*_{\Lambda\smallsetminus a}(\mathbb{R}^4) \times S^3 = 4(|\Lambda| - 1 - 1) - 1 + 3$$

thus is 0. Note that $|\Lambda| \geq 3$ is necessary for the inequality to make sense.

Case (4): We repeat the reasoning from the previous case, which is now easier as we do not even need the S^3 factor and $\omega(\Gamma_\Lambda)$ is a pullback of a form on a space of 4 dimensions smaller. ⊠

Lemma 7.5. When $|\Lambda| = 2$ and $\infty \in \Lambda$

$$\pi^\Lambda_* \omega(\Gamma) = 0$$

Proof. Let $\Lambda = \{j, \infty\}$. In that case the graph $\Gamma/(N \smallsetminus \Lambda)$ consists of two points j and say i, and valence-of-j edges between them. $\omega(\Gamma/(N \smallsetminus \Lambda))$ is then (up to sign depending on orientation of the edges) a wedge of pullbacks by the same map $\varphi'\psi_{ji}$ of the same volume form on S^3 so is 0. ⊠

Lemma 7.6. When $|\Lambda| = 2$, $\infty \notin \Lambda$ (and vertices of Λ are connected by an edge), then

$$\pi^\Lambda_* \omega(\Gamma) = (-1)^{\text{label}(\Lambda\text{-edge})} I(\Gamma/\Lambda)$$

Remark 7.7. Recall the formula

$$\delta(\Gamma) = \sum_{i\colon \text{edges of } \Gamma} (-1)^{\text{label}(i)-1} \Gamma/i$$

and observe the difference in signs. This the reason for the chain map being up to sign.

Proof of Lemma 7.6. Let $\lambda := \text{label}(\Lambda\text{-edge})$. By Proposition 7.2

$$\omega(\Gamma) = (-1)^{\lambda-1} p_2^* \omega(\Gamma_\Lambda) \wedge p_1^* \omega(\Gamma/\Lambda)$$

We compute

$$\begin{aligned}
\pi_*^\Lambda \omega(\Gamma) &= \pm \int_{E\overline{C}_{v,\Lambda}(\pi)} p_1^*\omega(\Gamma/\Lambda) \int_{\overline{C}_2^*(\mathbb{R}^4)} p_2^*\omega(\Gamma_\Lambda) \\
&= \pm \int_{E\overline{C}_{v,\Lambda}(\pi)} p_1^*\omega(\Gamma/\Lambda) \int_{\overline{C}_2^*(\mathbb{R}^4)} p_2^*\psi_\Lambda^*\varphi'^*\mathrm{vol}_{S^3} \\
&= \pm \int_{E\overline{C}_{v,\Lambda}(\pi)} p_1^*\omega(\Gamma/\Lambda) \int_{S^3} \mathrm{vol}_{S^3} \\
&= \pm \int_{E\overline{C}_{v,\Lambda}(\pi)} p_1^*\omega(\Gamma/\Lambda) \\
&= \pm I(\Gamma/\Lambda)
\end{aligned}$$

Where the sign is determined by the orientation induced on $\overline{S}_\Lambda$ from $E\overline{C}_v(\pi)$ and the sign coming from permutations in decomposition of $\omega(\Gamma)$. The sign from permutation we already know: it is $(-1)^{\lambda-1}$.

We only need to compute the induced orientation on the stratum $\overline{S}_\Lambda$. It was created by the blowup along $\{x_a = x_b\}$, where $\Lambda = \{a, b\}$. Neighborhood of a generic point is diffeomorphic to

$$\begin{aligned}
&(\partial\mathcal{B}\ell(\mathbb{R}^4 \times \mathbb{R}^4, \Delta)) \times (\mathbb{R}^4)^{v-2} \\
&\simeq \partial\mathcal{B}\ell(\mathbb{R}^4, 0) \times \mathbb{R}^4 \times (\mathbb{R}^4)^{v-2} \\
&\simeq S^3 \times \mathbb{R}^4 \times (\mathbb{R}^4)^{v-2}
\end{aligned}$$

The order of $\mathbb{R}^4$ factors does not matter as they are of even dimension. The induced orientation on S^3 is negative to the usual one (the one that paired with vol_{S^3} returns 1) since the outward normal vector to $u \in \partial\mathcal{B}\ell(\mathbb{R}^4, 0)$ is $-u$. Hence the final sign is

$$(-1)^\lambda$$

$\boxtimes$

Proof of Theorem 6.1.
<u>(1) $dI(\Gamma) = (-1)^{k+l+1} I(\delta\Gamma)$</u>

$$\begin{aligned}
dI(\Gamma) &= \overline{C}_v(\pi)_* d\omega(\Gamma) + (-1)^{k+l}\overline{C}_v(\pi)_*^\partial \omega(\Gamma) && \text{by Stokes} \\
&= (-1)^{k+l}\overline{C}_v(\pi)_*^\partial \omega(\Gamma) && \omega(\Gamma) \text{ is closed} \\
&= (-1)^{k+l} \sum_{\substack{\Lambda \subset N \cup \infty \\ |\Lambda| \geq 2}} \pi_*^\Lambda \omega(\Gamma) && \text{by definition} \\
&= (-1)^{k+l} \sum_{\substack{\Lambda \subset N \cup \infty \\ |\Lambda| = 2}} \pi_*^\Lambda \omega(\Gamma) && \text{by Lemma 7.4} \\
&= (-1)^{k+l} \sum_{\substack{\Lambda \subset N \\ |\Lambda| = 2}} \pi_*^\Lambda \omega(\Gamma) && \text{by Lemma 7.5} \\
&= (-1)^{k+l+1} = I(\delta\Gamma) && \text{by Lemma 7.6}
\end{aligned}$$

(2) I_* is independent of ω

Consider the bundle

$$\overline{C}_v(I \times \pi)\colon I \times E\overline{C}_v(\pi) \longrightarrow I \times B$$

The framing τ_E can be extended by the product structure. Consider two propagators ω_0 and ω_1 on the ends: $\{0,1\} \times E\overline{C}_v(\pi)$. By Proposition 5.7 there exists a propagator ω for τ_E extended on $I \times E\overline{C}_v(\pi)$ that extends both ω_0 and ω_1. Now consider sole B as a base of the bundle

$$\overline{C}_v(\pi)^I\colon I \times E\overline{C}_v(\pi)$$

and integrate ω along the fiber. As in the proof of (1) we use the Stokes theorem:

$$\begin{aligned} d\overline{C}_v(\pi)^I_*\omega(\Gamma) &= \pm\overline{C}_v(\pi)^{I\partial} \\ &= \pm\Big(\overline{C}_v(\pi)_*\omega_1(\Gamma) - \overline{C}_v(\pi)_*\omega_0(\Gamma) \pm (I \times B \to B)_*\overline{C}_v(\pi)^\partial_*\omega(\Gamma)\Big) \end{aligned}$$

The first $\pm$ comes from Stokes theorem, the second from orientation of the boundary of $I \times \overline{C}_v(\pi)$, which will not be important. Pair the previous equation with a δ-cocycle γ and use (1) on the third term:

$$\begin{aligned} dI_\omega(\gamma) &= \pm\Big(I_{\omega_1}(\gamma) - I_{\omega_2}(\gamma) \pm \int_I I_\omega(\delta\gamma)\Big) \\ &= \pm\Big(I_{\omega_1}(\gamma) - I_{\omega_2}(\gamma)\Big) \end{aligned}$$

Hence $I_{\omega_1 *}$ and $I_{\omega_0 *}$ differ up to sign by an exact form thus are equal in cohomology.

(3) I_* is independent of the edge-orientations

Swapping the orientation of one edge gives a diffeomorphism: $\varsigma\colon E\overline{C}_2(\pi) \longrightarrow E\overline{C}_2(\pi)$ that swaps 1 and 2, thus a new propagator for this edge. This new propagator agrees with the old one on the vertical boundary since

$$\begin{tikzcd} E\overline{C}_2(\pi) \arrow[r, "\varsigma"] \arrow[d, "p(\tau)"'] & E\overline{C}_2(\pi) \arrow[d, "p(\tau)"] \\ S^3 \arrow[r, "\iota"'] & S^3 \end{tikzcd}$$

commutes by Proposition 5.1. Observe that the proofs of (1) and (2) work even if we pick different propagators for each edge. That concludes the proof.

(4) I_* is invariant with respect to homotopy of τ_E

Consider the homotopy between framings as a framing on $I \times E\overline{C}_2(\pi)$. We repeat the proof of (2).

(5) I_* is natural with respect to bundle maps

For a map $B \longrightarrow B'$ we have the following commutative diagram:

$$\begin{tikzcd} E\overline{C}_v(\pi) \arrow[r, "\tilde{f}"] \arrow[d] & E\overline{C}_v(\pi') \arrow[d] \\ E \arrow[r] \arrow[d, "\pi"'] & E' \arrow[d, "\pi'"] \\ B \arrow[r, "f"'] & B' \end{tikzcd}$$

which gives rise to:

$$\begin{tikzcd}
 & P_k H^l(\mathcal{G}, \mathbb{Q}) \arrow[dl] \arrow[dr] & \\
H^{3e}_{\mathrm{dR}}(E\overline{C}_v(\pi)) \arrow[d, "\overline{C}_n(\pi)_*"'] & & H^{3e}_{\mathrm{dR}}(E\overline{C}_v(\pi')) \arrow[ll, "\tilde{f}^*"'] \arrow[d, "\overline{C}_n(\pi')_*"] \\
H^{k+l}_{\mathrm{dR}}(B) & & H^{3e-4v}_{\mathrm{dR}}(B') \arrow[ll, "f^*"']
\end{tikzcd}$$

The upper triangle *may not commute*, but $\tilde{f}^*\omega(\Gamma)$ is a wedge different propagators hence its integral can be identified with the one of $\omega(\Gamma)$ by Theorem 6.1(2). $\boxtimes$

8. Results

Recall 'the $(1, \ldots, 1)$ class'

$$\tilde{\zeta}_k = \frac{1}{(2k)!(3k)!}([\cdot] \otimes \mathrm{id})\zeta_k \in P_k H^0(\mathcal{G}, \mathcal{A}_k)$$

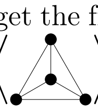
Effectively, we sum over all graphs that span $\mathcal{A}_k$, which in our case is redundant as $\mathcal{A}_2$ is one-dimensional. By Theorem 6.1 it gives a cohomology class of B:

$$I_*(\tilde{\zeta}_k) \in H^{k+l}(B; \mathcal{A}_k \otimes \mathbb{R})$$

When $\dim B = k$ it can be paired with the fundamental class of B to obtain an element of $\mathcal{A}_k \otimes \mathbb{R}$.

If we specialize $B = S^2$, $k = 2$ we obtain a linear map from the classifying of space of framed (D^4, ∂)-bundles

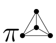
$$\mathcal{Z}_2 \colon \pi_2 \,\mathrm{B}\,\mathrm{Diff}(D^4, \partial) \otimes \mathbb{R} \longrightarrow \mathcal{A}_2 \otimes \mathbb{R}$$

We in fact forget the framing in classifying space, due to discussion in Section 3. Recall that $\mathcal{A}_2 = \mathbb{Q}\left\langle\right.$  $\left.\right\rangle$.

Theorem 8.1. The map $\mathcal{Z}_2$ is non-zero i.e., there exists a bundle π $\in \pi_2 \,\mathrm{B}\,\mathrm{Diff}(D^4; \partial)$ such that

$\mathcal{Z}_2(\pi$$) =$ 

hence $\dim \pi_2(\mathrm{B}\,\mathrm{Diff}(D^4; \partial) \otimes \mathbb{R} \geq 1$.

In [Wat19] and [Wat23] Watanabe describes a very concrete construction of the bundle π. Unfortunately, it is beyond the scope of this text.

Corollary 8.2. Since $\pi_k \,\mathrm{B}\, G \simeq \pi_{k-1} G$ we see that

$$\pi_1 \,\mathrm{Diff}(D^4, \partial) \otimes \mathbb{R} \geq 1$$

Finally, recall the splitting:

$$\mathrm{Diff}(S^n) \simeq \mathrm{Diff}(D^n, \partial) \times O(n+1)$$

we obtain the main result

$$\dim \pi_1 \,\mathrm{Diff}(S^4) \otimes \mathbb{R} \geq 1$$

Remark 8.3. It is known that $\mathrm{Top}(D^n, \partial)$ (group of homeomorphisms that fix the boundary pointwise) is contractible by the Alexander trick. Namely, for any $f \in \mathrm{Top}(D^n, \partial)$ we have an isotopy $f_t \colon f \sim \mathrm{id}$, that gives rise to the deformation retraction:

$$H(t, f) = f_t \in \mathrm{Top}(D^n, \partial) \qquad f_t(x) = \begin{cases} tf(x/t) & \text{for } 0 \leq ||x|| \leq t \\ t & \text{for } t \leq ||x|| \leq 1 \end{cases}$$

Thus Watanabe's bundle π is topologically trivial. Hence the name *exotic*.

A. Search for propagators for $S^2 \times S^2$

A propagator for a 4-manifold M is a 3-cochain in

$$H^3(\overline{C}_2(M, *), \partial) \simeq H^3(\mathring{M} \times \mathring{M} \smallsetminus \Delta, \partial)$$

that can be expressed as a pullback of the volume form by the Gauss map. However we may forget the second condition for a while and think of candidate-propagators. Let $M = S^2 \times S^2$. We heavily employ the $C_n(-) \longrightarrow \overline{C}_n(-)$ homotopy equivalence, throughout this section and the whole Appendix.

Proposition A.1. The manifold $\mathring{M}$ is parallelizable.

Proof. $\mathring{M}$ is homotopy equivalent to $S^2 \times S^2$. The tangent bundle $T\mathring{M}$ restricts under this homotopy equivalence to as sum $\mathbb{R}^2 \oplus TS^2$, where $\mathbb{R}^2$ is the trivial bundle, over each S^2. By the classification of bundles over the sphere[♠♠♠♠] this bundle is trivial. ⊠

Corollary A.2. $\nu\Delta$ is a trivial bundle.

Proof. It is a fiberwise restriction $\mathbb{R}^4 \longrightarrow D^4$ of the normal bundle to the diagonal which isomorphic to the tangent bundle. ⊠

A.1. Mayer–Vietoris

In this subsection all cohomologies are over $\mathbb{Z}$.

Consider the decoposition

$$\begin{aligned} \mathring{M} \times \mathring{M} &= (\mathring{M} \times \mathring{M} \smallsetminus \Delta) \cup \nu\Delta \\ &\qquad (\mathring{M} \times \mathring{M} \smallsetminus \Delta) \cap \nu\Delta \simeq \mathring{M} \times S^3 \end{aligned}$$

and its induced Mayer–Vietoris sequence:

$$\ldots \longrightarrow H^i(\mathring{M} \times \mathring{M}) \longrightarrow H^i(\nu\Delta) \times H^i(\mathring{M} \times \mathring{M} \smallsetminus \Delta) \longrightarrow H^i(\nu\Delta \cap (\mathring{M} \times \mathring{M} \smallsetminus \Delta)) \longrightarrow \ldots$$

Since $\nu\Delta$ is trivial:

$$\begin{aligned} &\nu\Delta \simeq \mathring{M} \times D^4 \\ &\nu\Delta \cap (\mathring{M} \times \mathring{M} \smallsetminus \Delta) \simeq M \times S^3 \end{aligned}$$

By the homotopy equivalence we compute cohomologies of M:

$$S^2 \vee S^2 \hookrightarrow S^2 \times S^2 \smallsetminus *$$

[♠♠♠♠] See for example [Hir].

$$H^i(\mathring{M}) = H^i(S^2 \vee S^2) = \mathbb{Z},\ 0,\ \mathbb{Z}^2,\ 0\dots$$

By the Künneth formula:

$$H^i(\mathring{M} \times \mathring{M}) = \mathbb{Z},\ 0,\ \mathbb{Z}^4,\ 0,\ \mathbb{Z}^4,\ 0\dots$$
$$H^i(\mathring{M} \times S^3) = \mathbb{Z},\ 0,\ \mathbb{Z}^2,\ \mathbb{Z},\ 0,\ \mathbb{Z}^2\dots$$

The Mayer–Vietoris sequence yields then:

$$
\begin{tikzcd}
 & H^i(\mathring{M} \times \mathring{M}) & H^i(\Delta \times D^4) \times H^i(\mathring{M} \times \mathring{M} \setminus \Delta) & H^i(\Delta \times S^3) & \\
0 & \mathbb{Z} \arrow[r] & \mathbb{Z} \times \mathbb{Z} \arrow[r] & \mathbb{Z} \arrow[dll] & \\
1 & 0 \arrow[r] & 0 \times \_\_ \arrow[r] & 0 \arrow[dll] & \\
2 & \mathbb{Z}^4 \arrow[r] & \mathbb{Z}^2 \times \_\_ \arrow[r] & \mathbb{Z}^2 \arrow[dll] & \\
3 & 0 \arrow[r] & 0 \times \_\_ \arrow[r] & \mathbb{Z} \arrow[dll] & \\
4 & \mathbb{Z}^4 \arrow[r] & 0 \times \_\_ \arrow[r] & 0 \arrow[dll] & \\
5 & 0 \arrow[r] & 0 \times \_\_ \arrow[r] & \mathbb{Z}^2 \arrow[r] & 0
\end{tikzcd}
$$

We see immediately see that

$$H^1(\overline{C}_2(M,*)) = 0$$
$$H^2(\overline{C}_2(M,*)) = \mathbb{Z}^4$$
$$H^5(\overline{C}_2(M,*)) = \mathbb{Z}^2$$

Then for other groups there are two options:

$$\begin{cases} H^3(\overline{C}_2(M,*)) = 0 \\ H^4(\overline{C}_2(M,*)) = \mathbb{Z}^3 \end{cases} \quad \text{or} \quad \begin{cases} H^3(\overline{C}_2(M,*)) = \mathbb{Z} \\ H^4(\overline{C}_2(M,*)) = \mathbb{Z}^4 \end{cases}$$

So we need to understand the map

$$\delta^* : H^3(\Delta \times S^3) \longrightarrow H^4(M \times M)$$

It could be done by embedding $\mathring{M} \times \mathring{M}$ in $\mathbb{R}^{12}$ and performing calculations on differential forms – everything is given by a formula. The disadvantage is lengthy computations, the advantage would be having a concrete concrete candidate for the propagator. This work is in preparation.

A.2. Leray–Serre spectral sequence

We try to use some more advanced machinery to compute $H^3(\overline{C}_2(M,*),\mathbb{Q})$. In this subsection all cohomologies are over $\mathbb{Q}$. This is no loss of generality since we are interested in representing propagators as differential forms – classes of de Rham cohomology.

Theorem A.3 (Leray-Serre spectral sequence, cf. [Hatb])**.** For a fibration $F \longrightarrow E \longrightarrow B$ of path connected spaces, with B simply connected there is a first quadrant spectral sequence of algebras with

$$E_2^{p,q} = H^p(B; H^q(F)) = H^p(B) \otimes H^q(F)$$

converging to $H^{p+q}(E)$ i.e.

$$H^n(E) = \bigoplus_{p+q=n} E_\infty^{p,q}$$

In our situation, $\overline{C}_2(M,*)$ can be viewed as a bundle over $\mathring{M}$ with fiber $\mathring{M} \smallsetminus *$ i.e.:

$$\mathring{M} \smallsetminus * \longrightarrow \overline{C}_2(M,*) \longrightarrow \mathring{M}$$

Observe that $\mathring{M} \smallsetminus * \simeq_{\text{hty}} S^2 \vee S^2 \vee S^3$ hence

$$H^i(B) = \mathbb{Q},\ 0,\ \mathbb{Q}^2,\ 0\dots$$
$$H^i(F) = \mathbb{Q},\ 0,\ \mathbb{Q}^2,\ \mathbb{Q},\ 0\dots$$

The E_2-page look as follows:

$E_2^{p,q} =$

3	$\mathbb{Q}$	0	$\mathbb{Q}^2$	0
2	$\mathbb{Q}^2$	0	$\mathbb{Q}^4$	0
1	0	0	0	0
0	$\mathbb{Q}$	0	$\mathbb{Q}^2$	0
p \ q	0	1	2	3

so we need tu understand the same map $\mathbb{Q} = E_2^{3,0} \longrightarrow E_2^{2,2} = \mathbb{Q}^4$, but over $\mathbb{Q}$, which is not better in any way.

A.3. Homological computation

We can also perform geometric reasoning on homology classes and then translate the result to cohomology by the universal coefficients formula. All homologies are over $\mathbb{Q}$. Again, this yields no loss of generality since we are interested in de Rham cohomology.

Corollary A.4 (to the Universal Coefficient Theorem, see [Hat01])**.** For any topological space X with finitely generated homology groups over $\mathbb{Q}$ the following holds:

$$H^n(X;\mathbb{Q}) \simeq H_n(X;\mathbb{Q})$$

The Mayer–Vietoris sequence looks the same, just with reversed arrows.

$$
\begin{tikzcd}[column sep=large]
 & H_i(\Delta \times S^3) & H_i(\Delta \times D_4) \oplus H_i(\mathring{M} \times \mathring{M} \setminus \Delta) & H_i(\mathring{M} \times \mathring{M}) & \\
5 & \mathbb{Q}^2 \arrow[r] & 0 \oplus \_\_ \arrow[r] & 0 \arrow[dll, rounded corners] & \\
4 & 0 \arrow[r] & 0 \oplus \_\_ \arrow[r] & \mathbb{Q}^4 \arrow[dll, rounded corners] & \\
3 & \mathbb{Q} \arrow[r] & 0 \oplus \_\_ \arrow[r] & 0 \arrow[dll, rounded corners] & \\
2 & \mathbb{Q}^2 \arrow[r] & \mathbb{Q}^2 \oplus \_\_ \arrow[r] & \mathbb{Q}^4 \arrow[dll, rounded corners] & \\
1 & 0 \arrow[r] & 0 \oplus \_\_ \arrow[r] & 0 \arrow[dll, rounded corners] & \\
0 & \mathbb{Q} \arrow[r] & \mathbb{Q} \oplus \mathbb{Q} \arrow[r] & \mathbb{Q} \arrow[r] & 0
\end{tikzcd}
$$

We are interested in the map

$$\partial \colon \mathbb{Q}^4 \simeq H_4(\mathring{M} \times \mathring{M}) \longrightarrow H_3(\Delta \times S^3) \simeq \mathbb{Q}$$

We use the description provided in [Hat01, Section 2.2] to compute its image. Since $\mathring{M} \simeq_{hty} S^2 \vee S^2$, each class of $H_4(\mathring{M} \times \mathring{M})$ is represented by a product of 2-spheres. We denote these spheres by S_1, S_2, S_1', S_2', where S_1 and S_2 are classes in the first copy of $\mathring{M}$ in $\mathring{M} \times \mathring{M}$, while S_1' and S_2' of the second one. Then

$$H_4(\mathring{M} \times \mathring{M}) = \mathbb{Q}\big\langle [S_1 \times S_1'],\ [S_1 \times S_2'],\ [S_2 \times S_1'],\ [S_2 \times S_2'] \big\rangle$$

Let us examine the image of the first class: $[S_1 \times S_1']$. We first need to express it as sum of two chains, with boundary in $\nu\Delta$. Let

$$
\begin{aligned}
\Delta_1 &\coloneqq \{(x,x) \subset S_1 \times S_1'\} \\
\nu_1\Delta_1 &\coloneqq \{\text{tubular neighbourhood of } \Delta_1 \text{ in } S_1 \times S_1'\} \simeq \Delta_1 \times D^2
\end{aligned}
$$

then

$$[S_1 \times S_1'] = [\nu_1\Delta_1] + [S_1 \times S_1' \setminus \nu_1\Delta_1]$$

The boundary of $\nu_1\Delta_1$ is null-homologous in $S^3 \times \Delta$ since it does not contain the normal-to-Δ 3-sphere.

Let us look at $[S_1 \times S_2']$ then. Its intersection with Δ is only one point – the "wedging point" of $M \simeq S^2 \vee S^2$, let us call it α. In $C_2(M, *) = \mathring{M} \times \mathring{M} \setminus \Delta$, α is cut out hence we will actually looking at its neighbourhood in $[S_1 \times S_2']$, call it $D(\alpha)$ since it is a 4-disk when completed with α. We see that

$$[S_1 \times S_2'] = [D(\alpha)] + [S_1 \times S_2' \setminus D(\alpha)]$$

Since $D(\alpha)$ intersects Δ in one point only it can be homotoped to be normal to it. Thus $\delta[D(\alpha)]$ is the generator of $H_3(\Delta \times S^3)$ and hence

$$\partial \colon \mathbb{Q}^4 \simeq H_4(\mathring{M} \times \mathring{M}) \longrightarrow H_3(\Delta \times S^3) \simeq \mathbb{Q}$$

is onto. Therefore by exactness

$$H_3(C_2(M,*)) = 0$$

making no room for candidate-propagators.

B. Search for propagators for $\mathbb{C}\mathrm{P}^2$

B.1. Normal to the diagonal

To apply the Mayer–Vietoris sequence we need to prove that $\partial\nu\Delta$ in $\mathring{\mathbb{C}\mathrm{P}}^2 \times \mathring{\mathbb{C}\mathrm{P}}^2$ is trivial. We show that it is homologically trivial.

Observe that $\mathring{\mathbb{C}\mathrm{P}}^2$ is homotopy equivalent to S^2 hence $\partial\nu\Delta$ is homotopy equivalent to an S^3 bundle over S^2 that embeds into a 4-dimensional real vector bundle. We call the total space of this bundle X. X is a union of two copies of $D^2 \times S^3$ glued over ∂D^2 by a *clutching function*, see [Hir] for reference.

We write down the Mayer–Vietoris sequence for that decomposition:

$$\begin{aligned} X &= (D^2 \times S^3) \cup (D^2 \times S^3) \\ (D^2 \times S^3) \cap (D^2 \times S^3) &= S^1 \times S^3 \end{aligned}$$

The homologies are over $\mathbb{Z}$.

$$\begin{tikzcd}
 & H_i(S^1 \times S^3) & H_i(S^3) \oplus H_i(S^3) & H_i(X) & \\
5 & 0 \arrow[r] & 0 \arrow[r] & -- \arrow[dll, rounded corners] & \\
4 & \mathbb{Z} \arrow[r] & 0 \arrow[r] & -- \arrow[dll, rounded corners] & \\
3 & \mathbb{Z} \arrow[r] & \mathbb{Z}^2 \arrow[r] & -- \arrow[dll, rounded corners] & \\
2 & 0 \arrow[r] & 0 \arrow[r] & -- \arrow[dll, rounded corners] & \\
1 & \mathbb{Z} \arrow[r] & 0 \arrow[r] & -- \arrow[dll, rounded corners] & \\
0 & \mathbb{Z} \arrow[r] & \mathbb{Z}^2 \arrow[r] & \mathbb{Z} \arrow[r] & 0
\end{tikzcd}$$

We immediately see that:

$$\begin{aligned} H_5(X) &\simeq \mathbb{Z} \\ H_2(X) &\simeq 0 \end{aligned}$$

If $H_1(X)$ was non-zero, that is $\mathbb{Z}$, then $H_0(S^1 \times S^3) \longrightarrow H_0(S^3) \oplus H_0(S^3)$ would be 0 hence the next arrow would be an isomorphism between $\mathbb{Z}^2$ and $\mathbb{Z}$ which is impossible. We look at the third row:

$$\begin{array}{ccccccc} H_3(S^1 \times S^3) & \longrightarrow & H_3(S^3) \oplus H_3(S^3) & \longrightarrow & H_3(X) & & \\ (1) & \longmapsto & (1,1) & \longmapsto & (1-1) & = & (0) \\ & & (a,b) & \longmapsto & (a-b) & & \end{array}$$

Thus by exactness $H_3(X) \simeq \mathbb{Z}$ and, by exactness again, $H_4(X) \simeq 0$. Putting this all together we obtain:

$$H_i(X) \simeq \mathbb{Z},\ 0,\ \mathbb{Z},\ \mathbb{Z},\ 0,\ \mathbb{Z},\ 0,\ \ldots$$

while

$$H_i(S^2 \times S^3) \simeq \mathbb{Z},\ 0,\ \mathbb{Z},\ \mathbb{Z},\ 0,\ \mathbb{Z},\ 0,\ \ldots$$

is the same.

B.2. Finding the propagator by Mayer–Vietoris

We repeat the homological reasoning from Section A.3. In this subsection the coefficients are over $\mathbb{Q}$.

$$\begin{aligned} \mathring{\mathbb{C}\mathrm{P}}{}^2 \times \mathring{\mathbb{C}\mathrm{P}}{}^2 &\simeq C_2(\mathbb{C}\mathrm{P}^2, *) \cup \nu\Delta \\ C_2(\mathbb{C}\mathrm{P}^2, *) \cap \nu\Delta &\simeq \partial\nu\Delta \end{aligned}$$

$$\begin{aligned} H_i(\mathring{\mathbb{C}\mathrm{P}}{}^2) &\simeq \mathbb{Q},\ 0,\ \mathbb{Q},\ 0,\ \ldots \\ H_i(\mathring{\mathbb{C}\mathrm{P}}{}^2 \times \mathring{\mathbb{C}\mathrm{P}}{}^2) &\simeq \mathbb{Q},\ 0,\ \mathbb{Q}^2,\ 0,\ \mathbb{Q},\ 0,\ \ldots \\ H_i(\Delta \times S^3) &\simeq \mathbb{Q},\ 0,\ \mathbb{Q},\ \mathbb{Q},\ 0,\ \mathbb{Q},\ 0,\ \ldots \end{aligned}$$

$$\begin{tikzcd}
 & H_i(\partial\nu\Delta) & H_i(\Delta) \oplus H_i(C_2(\mathbb{C}\mathrm{P}^2, *)) & H_i(\mathring{\mathbb{C}\mathrm{P}}{}^2 \times \mathring{\mathbb{C}\mathrm{P}}{}^2) & \\
5 & \mathbb{Q} \arrow[r] & 0 \oplus __ \arrow[r] & 0 \arrow[dll] & \\
4 & 0 \arrow[r] & 0 \oplus __ \arrow[r] & \mathbb{Q} \arrow[dll] & \\
3 & \mathbb{Q} \arrow[r] & 0 \oplus __ \arrow[r] & 0 \arrow[dll] & \\
2 & \mathbb{Q} \arrow[r] & \mathbb{Q} \oplus __ \arrow[r] & \mathbb{Q}^2 \arrow[dll] & \\
1 & 0 \arrow[r] & 0 \oplus __ \arrow[r] & 0 \arrow[dll] & \\
0 & \mathbb{Q} \arrow[r] & \mathbb{Q}^2 \arrow[r] & \mathbb{Q} \arrow[r] & 0
\end{tikzcd}$$

Immediately,

$$\begin{aligned} H_5(C_2(\mathring{\mathbb{C}\mathrm{P}}{}^2)) &\simeq \mathbb{Q} \\ H_1(C_2(\mathring{\mathbb{C}\mathrm{P}}{}^2)) &\simeq 0 \end{aligned}$$

Again, the crucial arrow is

$$\partial\colon \mathbb{Q} \simeq H_4(\mathring{\mathbb{C}\mathrm{P}}{}^2 \times \mathring{\mathbb{C}\mathrm{P}}{}^2) \longrightarrow H_3(\partial\nu\Delta) \simeq \mathbb{Q}$$

The only generator of $H_4(\mathring{\mathbb{C}P}^2 \times \mathring{\mathbb{C}P}^2)$ is the class $[S^2 \times S^2]$. This is a diagonal class, like $[S_1 \times S_1']$ previously. We need to compute $\partial\nu_1\Delta_1$ – the boundary of the thickened diagonal in $S^2 \times S^2$. The Euler class of $\nu_1\Delta_1$ is, by the isomorphism to the tangent bundle, equal to $\chi(S^2) = 2$. By the classification of bundles over S^2 $\partial\nu_1\Delta_1$ is the Hopf bundle

$$S^1 \hookrightarrow S^3 \longrightarrow S^2$$

hence represents a non-zero class in $H_3(\partial\nu\Delta)$. Thus, again, we have found no room for a candidate-propagator.

C. Propagators for $(S^1)^4$

The computation of Section A.1 produces results when $\pi_1\mathring{X}$ is nontrivial. We compute $H^3C_2((S^1)^4; *)$ with the Mayer–Vietoris sequence.

$$\begin{aligned}(\mathring{S^1})^4 \times (\mathring{S^1})^4 &= \nu\Delta \cup C_2((\mathring{S^1})^4)\\ &= \nu\Delta \cap C_2(\mathring{S^1})^4 = \Delta \times S^3\end{aligned}$$

Relevant cohomologies of these spaces are as follows.

$$\begin{aligned}&H^*((\mathring{S^1})^4) \simeq H^*((S^1 \times S^1) \vee (S^1 \times S^1)) \simeq \mathbb{Z},\ \mathbb{Z}^4,\ \mathbb{Z}^2,\ 0,\ \ldots\\ &H^3((\mathring{S^1})^4 \times (\mathring{S^1})^4) \simeq (\mathbb{Z}^4 \otimes \mathbb{Z}^2) \oplus (\mathbb{Z}^2 \otimes \mathbb{Z}^4) \simeq \mathbb{Z}^{16}\\ &H^3(\nu\Delta) \simeq H^3(\Delta) \simeq 0\\ &H^4(\Delta \times S^3) \simeq \mathbb{Z}^4 \otimes \mathbb{Z} \simeq \mathbb{Z}^4\end{aligned}$$

As in the previous cases we are interested in the following part of the exact sequence:

$$H^4(\Delta \times S^3) \longrightarrow H^3((\mathring{S^1})^4 \times (\mathring{S^1})^4) \longrightarrow H^3(\nu\Delta) \oplus H^3C_2((\mathring{S^1})^4)$$

which is

$$\mathbb{Z}^4 \longrightarrow \mathbb{Z}^{16} \longrightarrow 0 \oplus \underbrace{H^3C_2((\mathring{S^1})^4)}_{\geqslant \mathbb{Z}^{12}}$$

Thus, there is plenty of candidate-propagators in this case.

D. General candidate-propagators

We generalize previous computations to every almost-parallelizable closed 4-manifold X, with $H^3(\mathring{X}) = 0$. Almost-parallelizability is necessary for triviality of the normal bundle to the diagonal. The part of the Mayer–Vietoris exact sequence we are interested in is

$$H_4(\mathring{X} \times \mathring{X}) \xrightarrow{\partial} H_3(\Delta \times S^3) \longrightarrow H_3(\Delta \times D^4) \oplus H^3C_2(X, *)$$

Triviality of $H^3(\mathring{X})$ guarantees that

$$\begin{aligned}&H^3(\Delta \times S^3) = \mathbb{Z}\langle [*] \otimes [S^3] \rangle \simeq \mathbb{Z}\\ &H^3(\Delta \times D^4) \simeq 0\end{aligned}$$

Thus we need to know if ∂ is onto. We already know that the *diagonal* classes are in $\ker \partial$ and *non-diagonal* classes are not. Therefore the map ∂ is not onto, and we have room for propagators, if

$$\begin{cases} \dim_{\mathbb{Q}} H^2(\mathring{X}, \mathbb{Q}) \leq 1 \\ \dim_{\mathbb{Q}} H^3(\mathring{X}, \mathbb{Q}) = 0 \end{cases}$$

Poincaré–Lefschetz duality translates this conditions into Betti numbers:

$$\begin{cases} H^2(\mathring{X}, \mathbb{Q}) \simeq H_2(\mathring{X}, \partial, \mathbb{Q}) \simeq H_2(X, \mathbb{Q}) \\ H^3(\mathring{X}, \mathbb{Q}) \simeq H_1(\mathring{X}, \partial, \mathbb{Q}) \simeq H_1(X, \mathbb{Q}) \end{cases}$$

So we need $b_1(X) = 0$ and $b_2(X) \leq 1$ for the existence of candidate-propagators. Moreover in Section B we show an example where $b_1 = b_2 = 0$ does not imply the existence of propagators.

Our result is slightly weaker from a criterion of Lin and Xie [LX23, Remark 1.1.14], who claim $b_1(X) = 0$, $b_2(X) \neq 0$ provides an obstruction to the existence of the propagator, without any parallelizability assumptions. To construct an actual propagator we would need to express it as a pullback of the Gauss map on the boundary and extend it to the whole space as in Section 5. We have not investigated this problem so far.

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